Colour Manipulation of Superposed Families of Curves

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When several families of closely grouped curves are superposed a complex pattern of interference often appears. In this paper we show how the use of colour can help to unravel these complexities by emphasising or suppressing selected components of the interference.

Received July 1986

1. INTRODUCTION

Families of curves in the plane can often be conveniently described as contours in the form $\{\Phi(x,y) = n : n \in N\}$, where N is some subset of the family of integers. Thus an equispaced family of concentric circles may be expressed as $\{(x^2+y^2)^{\frac{1}{2}} = n : n = 0, 1, 2, \ldots\}$ and a family of m lines radiating from the origin may be expressed as $\{(m/\pi)\tan^{-1}(y/x) = n : n = 0, 1, \ldots, m-1\}$. The superposition of such families of curves often causes interference to appear, and our main interest here lies in its observation and interpretation. The study of this interference involves a fascinating mixture through computer graphics of geometry, the physical properties of the eye, and the psychology of visual perception.

In a previous paper[2] the geometry of monochrome superposition of families was studied to explain both why and when we should see the various types of interference. Following this the usefulness of interference techniques in the visual interpretation of composition of flows in hydrodynamics was considered,[3] and later this work was extended by including the use of cine techniques.[4] Each of these papers was concerned with monochrome output for just two families of curves. As the number of families of curves increases so too does the complexity of the interference pattern observed, and our aim here is to demonstrate how colour might be used to interpret and unravel these complexities.

2. SUPERPOSITION TYPES

Of the types of superposition described by Bryngdahl,[1] just two are commonly observed in computer-drawn images. These are multiplicative and additive superposition.

If $d\Phi(x,y)$ represents the density at the point (x,y) for the family of curves $\{\Phi(x,y) = n : n \in N\}$, where $0 \leqslant d\Phi(x,y) \leqslant 1$, a convenient description of the density function for a family of curves drawn by a pen plotter or on a VDU is

$$\begin{cases} d\Phi(x,y) = 0 \text{ between the pen strokes,} \\ d\Phi(x,y) = 1 \text{ on the pen strokes.} \end{cases}$$

For most monochrome graph plotters or VDUs the density of ink or light on a pen stroke is essentially saturation density, so at points where two curves cross this density does not change. For this type of visual output a good approximation to the density function for two superposed families of curves described by the functions Φ and Ψ is given by $\min\{d\Phi(x,y) + d\Psi(x,y), 1\}$.

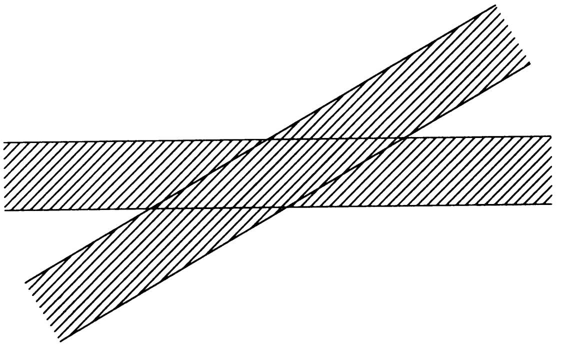

Figure 1. Multiplicative superposition.

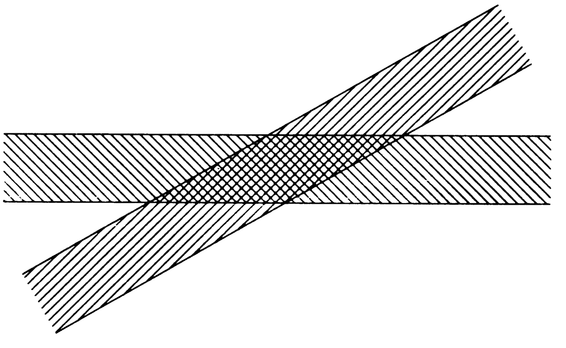

Figure 2. Additive superposition.

This is multiplicative superposition and is the situation illustrated in Fig. 1.

Some monochrome plotters allow the density of pen strokes to vary, and on these the second type of superposition can be displayed. If the density function for both of the families of curves described by the functions Φ and Ψ is set at 0 between the pen strokes and at no more than $\frac{1}{2}$ on the pen strokes, for the superposed families the density at the point (x,y) will be given by $d\Phi(x,y) + d\Psi(x,y)$. This is additive superposition and is the situation illustrated in Fig. 2. With additive superposition the intersection points are emphasised, and this is particularly useful with the cine techniques described in Ref. 4, where a flow is examined by observing the movement of these intersection points. Many colour plotters give essentially additive superposition in their output when only density is considered. Moreover the introduction of colour can emphasise even further the

Please note that the colour printing process has (a) changed some of the colours of the computer drawn images, thereby changing the emphasis, and (b) introduced extra interference (e.g. the herringbone effects in Fig. 18) as discussed by Firby[5].

intersection points, which can appear as a different colour to the underlying curves. The effectiveness of this use of colour to emphasise the intersection points can be seen in Figs. 3(*a*) and (*b*).

3. COLOUR SUPERPOSITION

Attempts to use a colour system to separate different components of a picture first of all bring out the fact that given a choice the eye has definite preferences for certain colours. This is illustrated in Fig. 4, in which three families of radial lines are superposed. One family is red, a second is blue and the third is green. The main visible components of this picture consist of three dipole effects each of the same density, but one is yellow, the second is cyan and the third is magenta.

These dipole effects are the interference patterns produced by each of the three pairs of families. For most people the yellow interference is more readily seen than the cyan interference, which in turn is much more readily seen than the magenta. Thus it is possible to distinguish separate components in a picture by manipulating the colour arrangement. However, the readiness of the eye to accept certain colours in preference to others suggests that a more effective use of colour would be to emphasise or suppress certain selected components.

Experiments with the colour output of the Dicomed D148C microfilm plotting system at the University of London Computer Centre suggest that yellow is one colour more readily accepted as dominant by the eye for this system, and therefore that a good way to get the interference between two families to stand out is to draw one of the families in red and the other in green. Figs 3(*a*) and (*b*), consisting of two superposed families of radial lines, illustrate the strength of the interference produced by this combination. In Fig. 3(*a*) the lack of symmetry is caused by the differing number of lines in the two families. In Fig. 3(*b*) several orders of interference are visible, the first-order interference being the central cross-point.

Several of the available colours are largely ignored by the eye in the presence of other colours and so may be effectively used to suppress part of a picture. Magenta is one of these, as can be seen in Fig. 4.

In practice this use of colour to emphasise or suppress selected interference components is rather more restricted than might at first appear. To understand these limitations we must look more closely at the combination of the colours in Dimfilm and also at the way the families of curves combine to give observable interference.

4. BUILDING COLOURS IN SUPERPOSED FAMILIES

The colour of a curve in one of our families can be specified by means of a triplet of values determining the amount of red, green and blue respectively in the required colour. Each of these values lies between 0 and 1. Thus $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ specify fully saturated red, green and blue respectively, $(\frac{1}{2},\frac{1}{2},0)$ specifies half-strength yellow and (x,x,x) specifies grey, with $(0,0,0)$ being black and $(1,1,1)$ being white.

The colour system is additive subject to a maximum value of 1 for each component, and thus the points at which n families of curves, with colours specified by (a_i, b_i, c_i) $(i = 1,2,\ldots,n)$ cross will have colour specified by $(\min\{\sum a_i, 1\}, \min\{\sum b_i, 1\}, \min\{\sum c_i, 1\})$. It can be seen that the more families of curves and the more colours we introduce the more likely it is that such crossing-points will appear grey, and it is through these crossing-points that we aim to emphasise or subdue the interference using colour. As well as this local assertion we can make the following global assertion. The more families and the more colours we introduce, the greater will be the tendency of the picture to appear a uniform monochrome and so the less effective will be the use of colour. As we shall demonstrate, these facts place severe limitations on the number of interference families which can be observed, and manipulated, within a picture.

Another point to note is that since the curves we consider do not cross themselves, if a picture contains n families then a point in the picture can have no more than n curves passing through it. Thus the colour of a point in the picture is determined by the combination of no more than n single colours, one from each of the families. This means that it is not usually possible to give controlled emphasis, through colour manipulation, to the interference caused by the interaction of two or more sets of interference whenever a single family is involved in the production of more than one of these sets of interference. For example, in Fig. 10 where we are considering the interference produced by three families of curves determined by Φ, Ψ, Γ we observe that the interference in region Z_2 (see Fig. 12) corresponds to the function $2\Phi - \Psi + \Gamma$, and that this can be thought of as the interaction of the interference corresponding to the function $\Phi - \Psi$ and the interference corresponding to the function $\Phi + \Gamma$. Although we can arrange the colours for the families Φ, Ψ, Γ so that the two sets of interference appear red and green, we cannot expect the interference for $2\Phi - \Psi - \Gamma$ to appear yellow as demonstrated in Fig. 17.

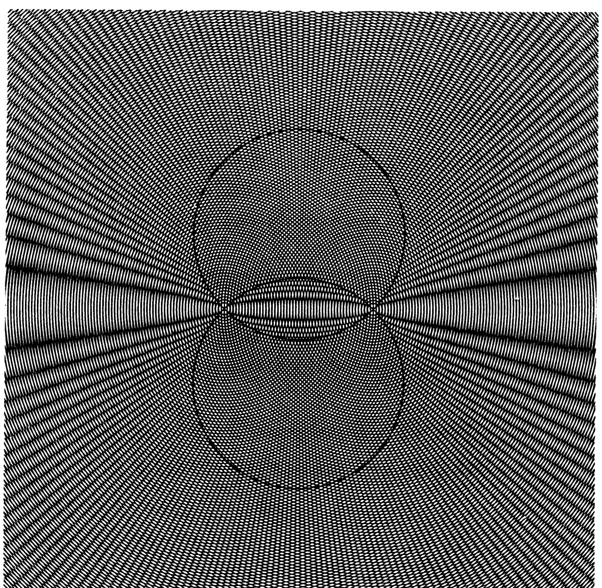

Figure 5. First-order interference for two families of equally spaced circles. The two circles describe the boundaries of the regions of interference.

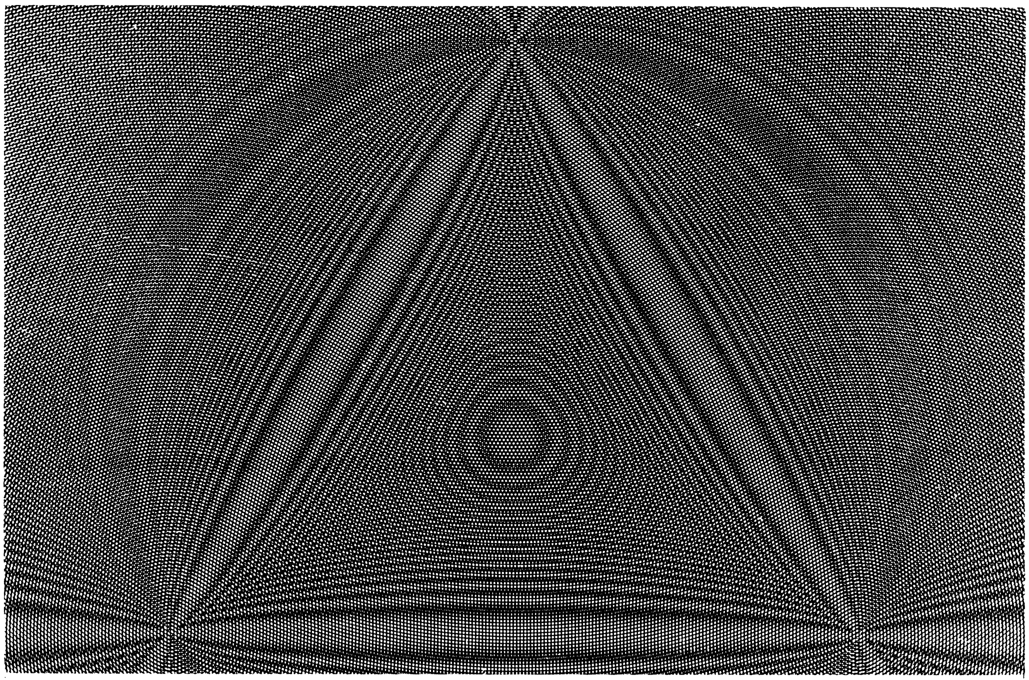

Figure 6. Interference for three families of equally spaced circles centred on the vertices of an equilateral triangle.

5. COLOUR MANIPULATION OF INTERFERENCE FAMILIES

In a previous paper[3] the complexity of the first-order interference pattern produced by two superposed families was analysed. When three or more families are superposed the complexity of the picture increases dramatically, often showing several orders of interference for each pair of families as well as the interference from more than two of the families. To illustrate this, and to show how the components of the picture can be recognised and then emphasised or suppressed selectively by the use of colour, we consider two examples of the superposition of three families of equispaced circles.

First of all, so that we can recognise the main components of the pictures, we recall the interference pattern for two superposed families of concentric circles shown in Fig. 5. As explained in Ref. 3, if the two families are described by the functions Φ and Ψ, outside the two overlapping circles is seen the family of interference fringes corresponding to the function $\Phi-\Psi$, and in the region of overlap of these two circles is seen the interference corresponding to $\Phi+\Psi$. In the remaining areas no interference is visible.

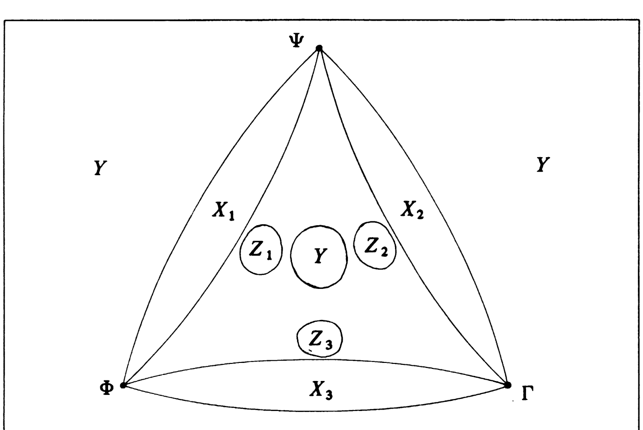

Figure 7. The main regions of interference visible in Fig. 6.

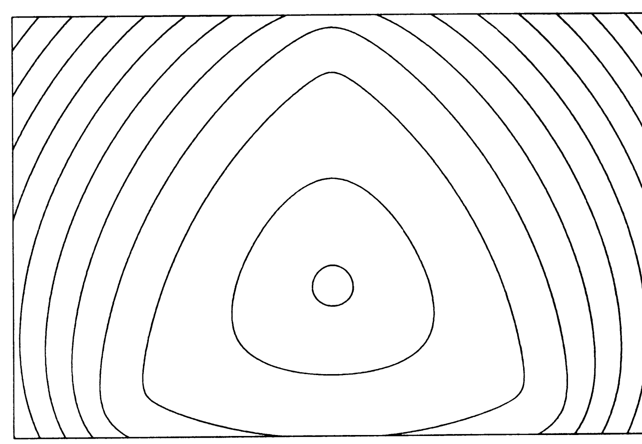

Figure 8. Contours for $\Phi+\Psi+\Gamma$ positioned as in Fig. 6.

In our first example of three-family superposition the families of circles are centred on the vertices of an equilateral triangle and the monochrome picture produced is shown in Fig. 6. If we suppose that the functions corresponding to these families are Φ, Ψ and Γ respectively then, referring to Fig. 7, the strongest components of the picture, appearing in regions X_1, X_2, X_3, are the interference fringes corresponding to $\Phi+\Psi$, $\Psi+\Gamma$ and $\Phi+\Gamma$ respectively. A little of the interference corresponding to $\Phi-\Psi$, $\Phi-\Gamma$ and $\Psi-\Gamma$ can be seen near the vertices outside the triangle, but the rest of the visible interference comes from three-family interaction. The strongest of the remaining visible components, appearing in the regions labelled Y, corresponds to $\Phi+\Psi+\Gamma$, as is illustrated by the contour map in Fig. 8. The very faint interference visible at positions Z_1, Z_2, Z_3 corresponds to $2\Phi+2\Psi+\Gamma$,

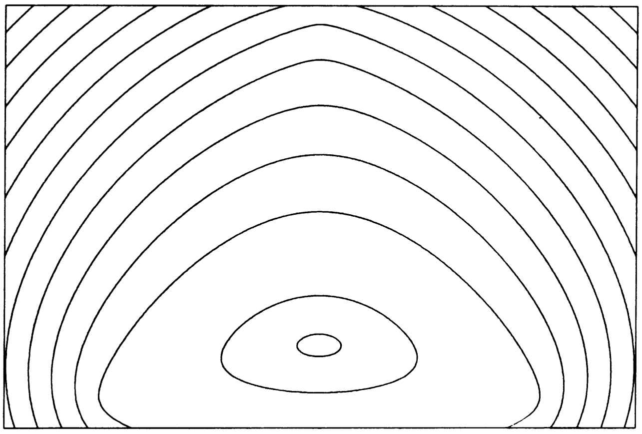

Figure 9. Contours for $2\Phi+\Psi+2\Gamma$ positioned as in Fig. 6.

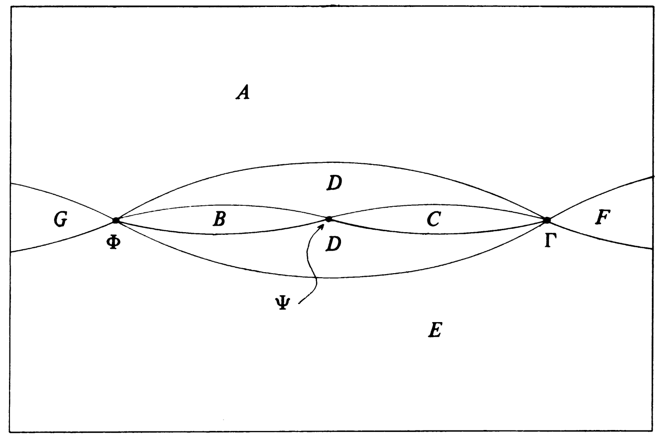

Figure 11. The main regions of interference visible in Fig. 10.

$\Phi+2\Psi+2\Gamma$ and $2\Phi+\Psi+2\Gamma$, respectively, as demonstrated by the contour map in Fig. 9. In fact these latter interference families correspond to the interaction of the interference family associated with $\Phi+\Psi+\Gamma$ with the interference families $\Phi+\Psi$, $\Psi+\Gamma$ and $\Phi+\Gamma$, respectively.

In our second example the three families of circles are centred on a line, and the monochrome picture produced is shown in Fig. 10. The interference formed by pairs of families can readily be seen and recognised. Thus referring to Fig. 11, which describes the approximate boundaries for this type of interference, in regions B and C are seen $\Phi+\Psi$ and $\Psi+\Gamma$ respectively, and in region D which includes regions B and C is seen $\Phi+\Gamma$. In regions A and E is seen a mixture of $\Phi-\Psi$, $\Phi-\Gamma$ and $\Psi-\Gamma$. In region F the dominant interference seen is that of $\Psi-\Gamma$ and in region G the dominant interference is that of $\Phi-\Psi$. The rest of the interference visible is caused by the interaction of the three families. Referring to Fig. 12, which again gives just approximate boundaries, in region X is seen the interference associated with $\Phi-\Psi+\Gamma$. In regions Y_1 and Y_2 appear $\Phi-2\Psi+2\Gamma$ and $2\Phi-2\Psi+\Gamma$ respectively. In regions Z_1 and Z_2 is the interference associated with $\Phi-\Psi+2\Gamma$ and $2\Phi-\Psi+\Gamma$ respectively. The latter interference is in fact caused by the interaction of the two sets of first-order interference fringes $\Phi+\Gamma$, $\Gamma-\Psi$ and $\Phi+\Gamma$, $\Phi-\Psi$ respectively. The sequence of contours shown in Fig. 13 demonstrate the validity of these assertions, showing the contours for each of the combinations of Φ, Ψ and Γ for which the interference is readily visible.

It is interesting to note that while in both of these examples the pattern of all possible combinations $l\Phi+m\Psi+n\Gamma$ $(l,m,n\in Z)$ exists within the picture, most are not visible. Also for those which are visible their contour pattern can be seen only over part of the picture. Thus for example one of the most basic combinations, that of $\Phi+\Psi+\Gamma$, can readily be seen in the first example. However, in the second example the pattern produced by this combination, described by the contour map in Fig. 14, is not sufficiently distinctive within this picture to be visible over any region.

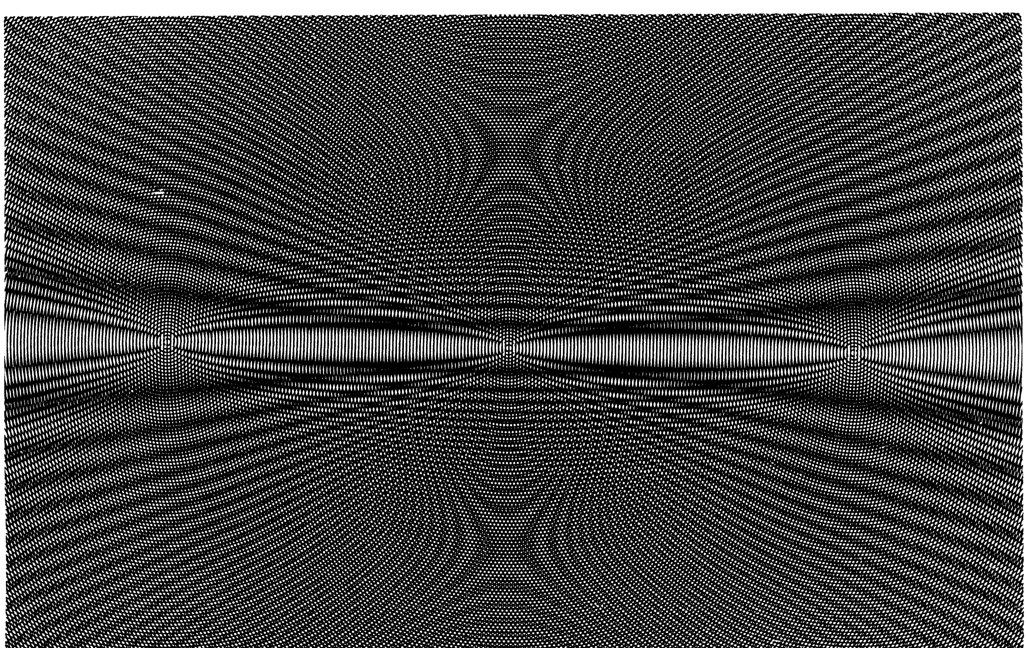

Figure 10. Interference for three families of equally spaced circles with centres positioned in a line.

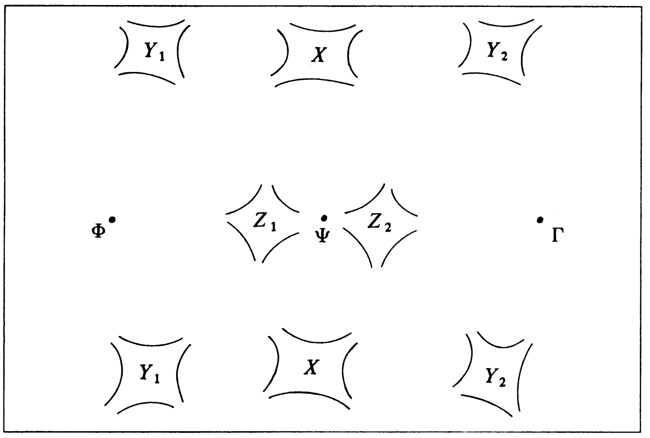

Figure 12. The main regions of higher-order interference visible in Fig. 10.

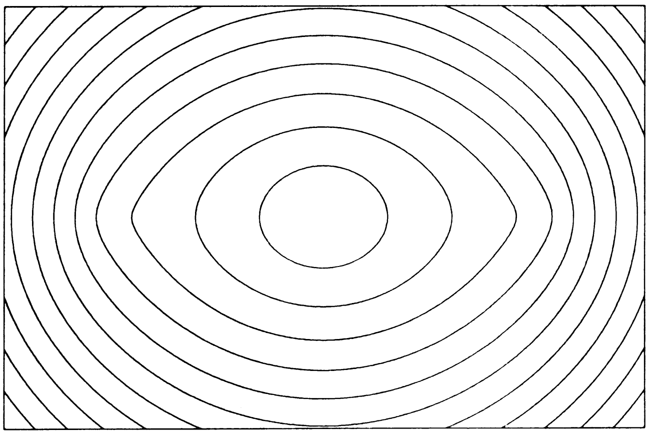

Figure 14. Contours for $\Phi+\Psi+\Gamma$ with centres positioned as in Fig. 10.

Finally, to see how colour can be used to distinguish, emphasise or suppress selected components in these examples, compare Figs 15 and 16 with Fig. 6 and Figs 17 and 18 with Fig. 10.

In Fig. 15 the families Φ, Ψ and Γ are shown in half-strength magenta $(\frac{1}{2}, 0, \frac{1}{2})$, cyan $(0, \frac{1}{2}, \frac{1}{2})$ and yellow $(\frac{1}{2}, \frac{1}{2}, 0)$ respectively. The $\Phi+\Psi$ interference thus appears as $(\frac{1}{2}, \frac{1}{2}, 1)$ (grey–blue), while $\Psi+\Gamma$ appears as $(\frac{1}{2}, 1, \frac{1}{2})$ (grey–green) and $\Phi+\Gamma$ appears as $(1, \frac{1}{2}, \frac{1}{2})$ (grey–red) and the three components of the interference pattern can be clearly distinguished.

In Fig. 16 an attempt has been made to emphasise the Φ, Ψ and Φ, Γ combinations while suppressing the Ψ, Γ combination. This has been done by colouring the Φ family $(\frac{1}{2}, 0, \frac{1}{2})$ (magenta), the Ψ family $(0, \frac{1}{2}, 0)$ (green), and the Γ family $(0, \frac{1}{2}, \frac{1}{2})$ (cyan). This produces $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ (grey) for the Φ, Ψ combination, $(0, 1, \frac{1}{2})$ (green–cyan) for the Ψ, Γ combination, and $(\frac{1}{2}, \frac{1}{2}, 1)$ (grey–blue) for the Φ, Γ combination with the effect shown.

In Fig. 17 the Φ, Ψ, Γ families are set at $(\frac{1}{2}, \frac{1}{2}, 0)$ (yellow), $(0, \frac{1}{2}, \frac{1}{2})$ (cyan) and $(\frac{1}{2}, 0, \frac{1}{2})$ (magenta) respectively. As in Fig. 15, this shows how the various components of interference appearing in the same or different regions may be distinguished using colour, but it also shows that

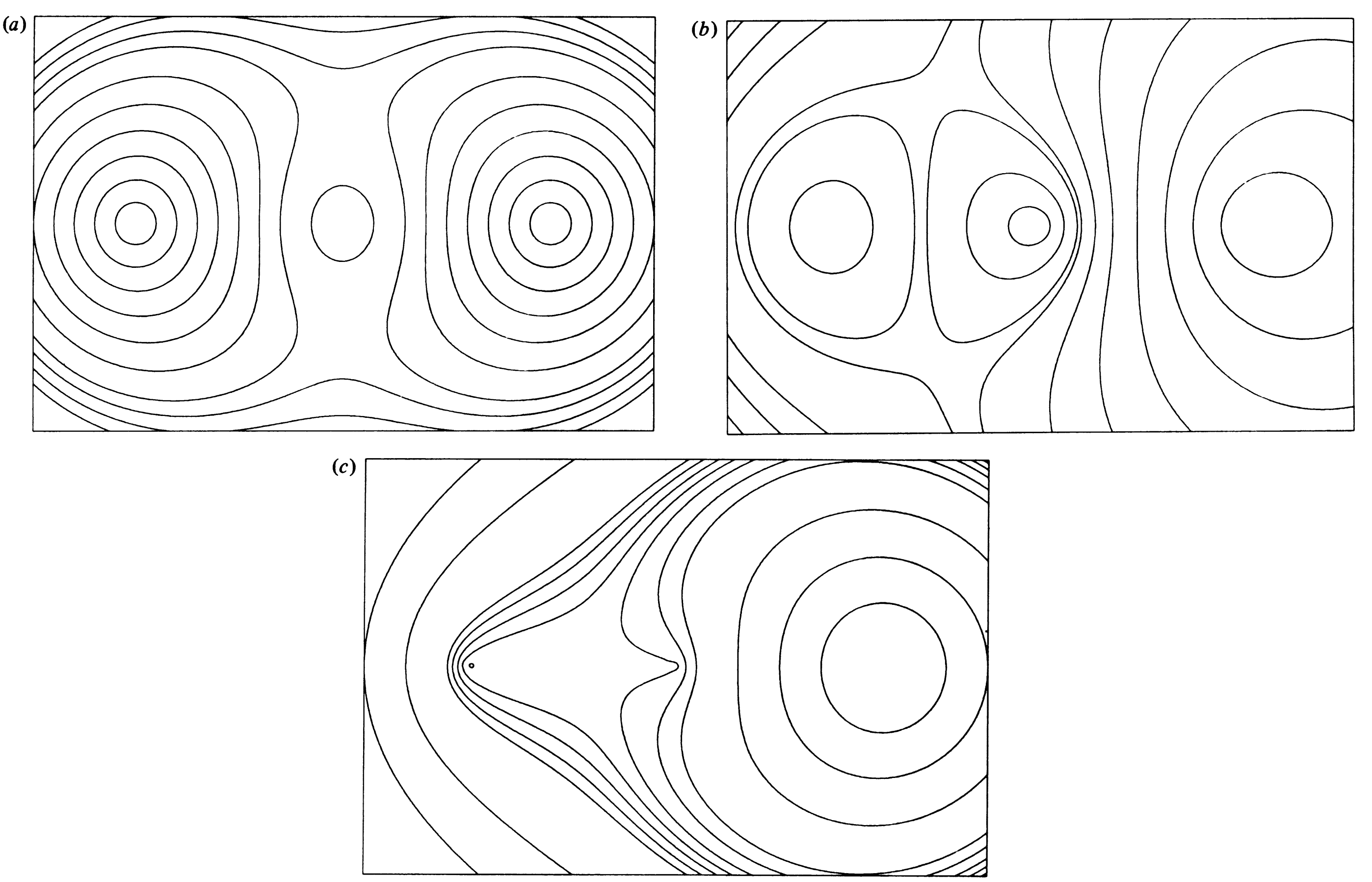

Figure 13. With the centres for Φ, Ψ, Γ positioned as in Fig. 10. (*a*) Contours for $\Phi-\Psi+\Gamma$. (*b*) Contours for $\Phi-2\Psi+2\Gamma$. (*c*) Contours for $\Phi-\Psi+2\Gamma$.

we cannot force a colour effect from the superposition of one interference family over a second interference family. With this colour combination we might expect the interference in region Z_2 of Fig. 12 to appear yellow, and therefore stand out, since it can be considered as the combination of $\Phi+\Gamma$ interference (grey–red) and $\Phi-\Psi$ interference (grey–green). As pointed out in Section 4, this does not happen because the colour at any point in the picture can be determined by at most three additions of the basic colours used, magenta, cyan and yellow, with the strengths given.

Finally in Fig. 18 the colours are set to emphasise the Φ, Ψ interaction and to suppress the Ψ, Γ and the Φ, Γ interaction. This also has the effect of suppressing all the three-family interference. The colours used here for the Φ, Ψ, Γ families are $(\frac{1}{2}, 0, 0)$ (red), $(\frac{1}{2}, 1, \frac{1}{2})$ (grey–green) and $(0, 0, \frac{1}{2})$ (blue) respectively.

6. CONCLUSION

Superposition of two or more families of curves can produce a complex picture containing interference of several orders associated with many of the possible combinations of the families. Interference associated with a particular combination is often visible over only a small region of the picture, being overwhelmed in the rest of the picture by the relative strength of the visible patterns there. Often the picture is further complicated by the appearance of more than one interference family over any given region.

Both the complexity of these pictures and the techniques used in their analysis are limited by several factors. As the order of the interference increases, or as the number of families producing a given area of interference increases, so does the contrast of the interference decrease. In colour pictures as the number of families of different colour producing the interference increases so does the interference itself become increasingly grey and less distinct. It follows that the higher the order of the interference and the larger the number of families combining to produce the interference, the less likely it is that it will be seen. It also follows that the higher the order of the interference or the larger the number of families combining to produce it, the less likely it is that the use of colour will be successful in emphasising this component.

A search through the contour maps for combinations of low order will therefore identify most of the components visible in a picture, and a careful use of colour can help to separate the specific interference families. It is possible to use colour to selectively emphasise or suppress particular interference components, but the inherent build-up of greys through the technique limits its use to low orders and to very small numbers of superposed families.

REFERENCES

1. O. Bryngdahl, Characteristics of superposed patterns in optics. *J. Opt. Soc. Am.* **66**, 87–94 (1976).
2. P. A. Firby, Angles of dominance for moiré fringes. *IMA Journal of Applied Mathematics* **33**, 303–321 (1984).
3. P. A. Firby and D. J. Stone, Interference patterns and 2-dimensional potential theory. *IMA Journal of Applied Mathematics* **33**, 291–301 (1984).
4. P. A. Firby and D. J. Stone, Interference in computer graphics. *Computer Graphics Forum* **4**, 209–216 (1985).
5. P. A. Firby, Interference in printing and textile manufacture. *Math. Spectrum* (to appear) (1987).