On the Consistency of Multi-valued Functions*

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Gallier[1] generalised the notion of consistency in studying the sufficient conditions for the existence of optimal fixpoints. Chen[2] introduced the notion of $\leqslant$-relatedness in investigating some fixpoints for the class of non-deterministic recursive programs. In this paper, it is proved that the notions of $\leqslant$-relatedness and consistency are equivalent for the class of multi-valued functions.

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1. INTRODUCTION

The notion of consistency was first proposed by Manna and Shamir[3] in their study of the optimal fixpoints for the class of deterministic recursive programs with domains that are flat partially ordered sets. Two single-valued functions are said to be consistent if, when they are defined, both are defined with the same value. They defined the optimal fixpoint as the greatest element of the set of consistent fixpoints, and showed that the optimal fixpoint is the best fixpoint semantics for the deterministic recursive programs.

Gallier[1] generalised their concept of consistency to arbitrary partially ordered sets. His generalisation is: two functions are said to be consistent if their least upper bound exists.

In his recent study of the various types of fixpoints for the class of non-deterministic recursive programs, Chen[2] introduced the notion of $\leqslant$-relatedness to extend Manna and Shamir's consistency to the class of multi-valued functions. This notion of $\leqslant$-relatedness plays a fundamental role in the study of the various fixpoint semantics of non-deterministic recursive programs. In this paper, we are going to show that the notions of $\leqslant$-relatedness and generalised consistency are equivalent for the class of multi-valued functions. Our result justifies Chen's[2] definition of optimal fixpoint for the class of non-deterministic recursive definitions, as his optimal fixpoint is identical to the greatest element of the set of $\leqslant$-related fixpoints.

2. PRELIMINARIES

In this section, we recall some basic definitions and establish our notation.

In this paper, D is used to denote a flat domain with bottom, ω, which stands for *undefined*. $SF(D)$ is used to denote the set of all single-valued functions from D into D, while $MF(D)$ is used to denote the set of all multi-valued functions from D into $2^D\backslash\varnothing$. For any $f \in MF(D)$ and $d \in D$, we use $\{f(d)\}$ to denote the set of all possible images of $f(d)$. $f(d)$ represents an element of $\{f(d)\}$. For any f and g of $MF(D)$, $f = g$ if $\{f(d)\} = \{g(d)\}$ for every $d \in D$. For any set S, lub (S) denotes its least upper bound if the latter exists.

As a reminder, a partially ordered set $(S, \leqslant)$ is said to be flat if there exists $a \in S$ such that $a \leqslant b$ for all $b \in S$, and no two distinct elements c and d of S for which $c \neq a$ and $d \neq a$ are related by $\leqslant$.

Now, let us state the less defined or equal ordering and the Egli[4] ordering.

Definitions

(1) Less Defined or Equal Ordering

$\leqslant$ on D is defined as $\omega \leqslant a$ and $a \leqslant a$ for every $a \in D$.

(2) Egli Ordering

$\leqslant$ on $(2^D\backslash\varnothing)$ is defined as for any S_1, $S_2 \in (2^D\backslash\varnothing)$, $S_1 \leqslant S_2$ if $\forall a \in S_1 \exists b \in S_2[a \leqslant b]$ and $\forall b \in S_2 \exists a \in S_1[a \leqslant b]$.

In fact, $(D, \leqslant)$ is a flat partially ordered set. In this paper, the same symbol $\leqslant$ is used for both the less defined or equal ordering and the Egli ordering, however, there should be no confusion as the context will clearly indicate which ordering is meant. Also, whenever appropriate, we may use the following equivalent form of Egli ordering:

$S_1 \leqslant S_2$ if and only if either $\omega \in S_1$ and $S_1 \subset (S_2 \cup \{\omega\})$, or $\omega \notin S_1$ and $S_1 = S_2$.

Now, we are going to extend the Egli ordering to $MF(D)$ as follows:

Definition

For any $f, g \in MF(D), f \leqslant g$ if $\{f(d)\} \leqslant \{g(d)\}$ for every $d \in D$.

Manna and Shamir's notion of consistency is defined as follows:

Definitions

(1) a and $b \in D$ are consistent, if a and b are identical when both are not ω.
(2) f and $g \in SF(D)$ are consistent, if $f(d)$ and $g(d)$ are consistent for every $d \in D$.
(3) Let S be a non-empty subset of $SF(D)$. S is said to be consistent, if f and g are consistent for every $f, g \in S$.

Gallier generalised Manna and Shamir's notion of consistency with the following definition:

Definition

Let $(P, \leqslant)$ be a partially ordered set. A non-empty subset Q of P is said to be consistent if lub (Q) exists.

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It is easy to show that this definition is equivalent to (3) of Manna and Shamir's definition when P is a flat partially ordered set.

Now, we are going to present our notion of $\leqslant$-relatedness.

Definitions

(1) A and $B \in (2^D \backslash \varnothing)$ are $\leqslant$-related, if $\forall a \in A \exists b \in B[a \leqslant b \vee b \leqslant a]$ and $\forall b \in B \exists a \in A[b \leqslant a \vee a \leqslant b]$.

(2) f and $g \in MF(D)$ are $\leqslant$-related if $\{f(d)\}$ and $\{g(d)\}$ are $\leqslant$-related for every $d \in D$.

(3) Let S be a non-empty subset of $MF(D)$. S is said to be $\leqslant$-related, if f and g are $\leqslant$-related for every f, $g \in S$.

It is obvious that (2) of the above definition is equivalent to (2) of Manna and Shamir's definition in the case in which $\{f(d)\}$ and $\{g(d)\}$ each contain a single element, for each $d \in D$.

3. MAIN RESULT

In this section, we are going to establish the equivalence between $\leqslant$-relatedness and generalised consistency for the class of multi-valued functions. As a reminder, D is used to denote a flat domain with ω as its bottom element throughout this paper.

First, let us present some properties of $\leqslant$-relatedness.

Lemma 1

Let $S \subset MF(D)$ be non-empty and $\leqslant$-related. For any $d \in D$ and $f \in S$, if $\omega \notin \{f(d)\}$, then $\{g(d)\} \leqslant \{f(d)\}$ for every $g \in S$.

Proof

For any $d \in D$ and $g \in S$, we have

(1) Assume $\omega \in \{g(d)\}$.
For any $b \in \{g(d)\}$, there exists $a \in \{f(d)\}$ such that $a \leqslant b$ or $b \leqslant a$. If b is not ω, then b is a because $\omega \notin \{f(d)\}$. Therefore, $(\{g(d)\} \backslash \{\omega\}) \subset \{f(d)\}$. Thus, $\{g(d)\} \leqslant \{f(d)\}$.

(2) Assume $\omega \notin \{g(d)\}$.
For any $a \in \{f(d)\}$, there exists $b \in \{g(d)\}$ such that $a \leqslant b$ or $b \leqslant a$. Since $\omega \notin (\{f(d)\} \cup \{g(d)\})$, a is b. Therefore, $\{f(d)\} \subset \{g(d)\}$. Similarly, it can be proved that $\{g(d)\} \subset \{f(d)\}$. Therefore, $\{g(d)\} \leqslant \{f(d)\}$. ■

By Lemma 1, we have the following property:

Lemma 2

Let $S \subset MF(D)$ be non-empty and $\leqslant$-related. For any f, $g \in S$ and $d \in D$, if $\omega \notin (\{f(d)\} \cup \{g(d)\})$ then $\{f(d)\} = \{g(d)\}$.

Proof

It follows immediately after Lemma 1. ■

Before we show the equivalence between the existence of least upper bound and $\leqslant$-relatedness, we need the following lemma.

Lemma 3

Let f, g, h be any elements of $MF(D)$. If $g \leqslant f$ and $h \leqslant f$, then g and h are $\leqslant$-related.

Proof

For any $d \in D$, we have

(1) Assume $\omega \in \{f(d)\}$.
It follows from the definition that $\omega \in \{g(d)\}$ and $\omega \in \{h(d)\}$. Obviously, $\{g(d)\}$ and $\{h(d)\}$ are $\leqslant$-related.

(2) Assume $\omega \notin \{f(d)\}$.
(i) Assume $\omega \notin (\{g(d)\} \cup \{h(d)\})$.
Since $\{g(d)\} = \{f(d)\} = \{h(d)\}$, $\{g(d)\}$ and $\{h(d)\}$ are obviously $\leqslant$-related.
(ii) Assume $\omega \in (\{g(d)\} \cup \{h(d)\})$.
If $\omega \in \{g(d)\}$ and $\omega \in \{h(d)\}$, obviously $\{g(d)\}$ and $\{h(d)\}$ are $\leqslant$-related. Without loss of generality, it is sufficient to consider the case that $\omega \in \{g(d)\}$ and $\omega \notin \{h(d)\}$. Since $\{h(d)\} = \{f(d)\}$, we have $\{g(d)\} \subset (\{h(d)\} \cup \{\omega\})$. Obviously, $\forall a \in \{g(d)\} \exists b \in \{h(d)\}[a \leqslant b]$. However, $\omega \in \{g(d)\}$ implies $\forall b \in \{h(d)\} \exists a \in \{g(d)\}[a \leqslant b]$. Therefore, $\{g(d)\}$ and $\{h(d)\}$ are $\leqslant$-related. ■

Now, we are going to show that $\leqslant$-relatedness is a necessary and sufficient condition for the existence of the least upper bound.

Theorem 1

Let $S \subset MF(D)$ be non-empty. S is $\leqslant$-related if and only if lub(S) exists.

Proof

(1) Assume lub(S) exists.
For any f, $g \in S$, we have $f \leqslant$ lub(S) and $g \leqslant$ lub(S). By Lemma 3, f and g are $\leqslant$-related. Thus, S is $\leqslant$-related.

(2) Assume S is $\leqslant$-related.
Because of Lemma 2, we can construct f_s in such a way that for any $d \in D$, if S has a f such that $\omega \notin \{f(d)\}$, then $\{f_s(d)\} = \{f(d)\}$ else

$$\{f_s(d)\} = \cup_{f \in S}\{f(d)\}.$$

We are now going to show that f_s is actually the lub(S).

For any $d \in D$, if there exists $g \in S$ such that $\omega \notin \{g(d)\}$, it follows from the definition of f_s and Lemma 1 that $\{f(d)\} \leqslant \{g(d)\} = \{f_s(d)\}$ for every $f \in S$. Otherwise, it follows from the definition that for every $f \in S$, $\{f(d)\} \leqslant \{f_s(d)\}$ because $\{f(d)\} \subset \{f_s(d)\}$. Therefore, we have $f \leqslant f_s$ for every $f \in S$, that is, f_s is an upper bound of S.

Let g be any upper bound of S. For any $d \in D$, if there exists $f \in S$ such that $\omega \notin \{f(d)\}$, it is obvious that $\{f_s(d)\} = \{f(d)\} \leqslant \{g(d)\}$. Otherwise, it follows from the definition that $\{f(d)\} \subset (\{g(d)\} \cup \{\omega\})$, for all $f \in S$. Therefore,

$$\bigcup_{f \in S} \{f(d)\} \subset (\{g(d)\} \cup \{\omega\}).$$

By definition,

$$\{f_s(d)\} = \bigcup_{f \in S} \{f(d)\} \leqslant \{g(d)\},$$

as $\omega \in \cup_{f \in S}\{f(d)\}$.

Therefore, f_s is equal to lub(S). ■

Thus, we have proved that the notions of $\leqslant$-relatedness and generalised consistency are equivalent for the class of multi-valued functions with domains that are flat partially ordered sets.

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Announcements

31 MAY TO 2 JUNE 1991

First Language International Conference

in association with the University of Copenhagen, Elsinore, Denmark

Call for Papers

'Training, talent and experience': aspects of the teaching of translation and interpreting

The conference is planned for 31 May to 2 June 1991 at the LO Conference Centre, Elsinore, Denmark.

Within the topic of the Conference participants are invited to give sectional papers of 30 minutes. The organisers are primarily interested in questions of contents and principles rather than syllabus, as long as it is borne in mind that reference is made to application in teaching practices.

The organisers suggest the following sub-themes for papers.

- Art, craft (and luck) in translation and interpreting
- Interpreting and translation, similarities and differences
- Interpreting, consecutive
- Interpreting, simultaneous
- Translation and terminology
- Literary and technical translation
- Machine translation
- Translation critique and translation training
- Individual and institutional work

The organisers are willing to consider case studies as well as papers with a theoretical approach. Work in progress is encouraged. The organisers are open to suggestions for other themes, but also reserve the right to cancel any of the suggested sub-themes.

Submissions for papers, containing an informative title of up to ten words, and a one-page summary, should be sent to Professor Cay Dollerup. Papers will be screened and information about acceptance will be forwarded within two months. For the actual conference there will be a request for a five-line summary in order to make for easy overview of the contents.

For further information contact:

Professor Cay Dollerup, Centre for Translation Studies and Lexicography, University of Copenhagen, Njalsgade 80, DK-2300 Copenhagen S, Denmark.

22–24 APRIL 1991

International Conference on Achieving Quality in Software

Scuola Superiore di Studi Universitari e di Perfezionamento 'S. Anna', Pisa, Italy

The Conference

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Design approaches range from 'commonsense' practical, through semi-formal (in the sense of 'using forms') and formal, up to the 'fabulous', yet always promising algebraic techniques. All these approaches and tools need a way to be compared, possibly integrated and made practical to use.

As functional and performance requirements alone are no longer sufficient to make a software product acceptable, the multi-dimensional concept of quality requirements may be able to help software technology in a way before unseen.

Achieving software quality (how much of it? at what cost?) may be the unifying guideline for a variety of tools and methods, and may provide the criterion for their comparison, as well.

This conference has the ambitious goal of developing a clear picture of the state of the art in software quality and of stimulating proposals about incorporating, adjusting, evaluating and certifying quality attributes in software products.

All the related topics and problems are addressed, including:

- quality metrics
- design for inspectability
- design for testability
- program-proving versus testing
- V&V in concurrent and parallel systems
- certification
- significant case studies
- user and developer perspective of quality

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The Conference Proceedings will be edited and published by a major publishing house and will be distributed to the conference participants.

16–18 SEPTEMBER 1991

Third International Conference on Software Engineering for Real-time Systems

Cirencester

The Third International Conference on Software Engineering for Real-time Systems is to be held at the Royal Agricultural College, Cirencester from 16 to 18 September 1991.

The Conference is being organised by the Institution of Electrical Engineers (IEE). It will cover the theoretical developments and practical implementation of real-time software systems, including the application of multiprocessors and parallel computing.

For further information contact: Conference Services, IEE, Savoy Place, London WC2R 0BL, UK. Tel: 071-240-1871, ext. 222.