# Short Note

## A Note on the Efficiency of an Interval Routing Algorithm

**An interval routing algorithm for general networks has been proposed separately by Santoro and Khatib and van Leeuwen and Tan. It works optimally for various regular networks, and for the non-regular case it never chooses a route longer than twice the diameter of the network. Van Leeuwen and Tan posed the fundamental question whether there is an optimal interval routing algorithm for any arbitrary network. In this paper we prove the lower-bound result that for certain networks the interval routing algorithm chooses a route which is half as long as the diameter of this network. This answers the question in a negative way.**

*Received July 1989, revised February 1990*

### 1. Introduction

The routing problem for any network and any pair of nodes (called sender and destination) is to route messages along the shortest possible path from the sender node to the destination node.

Various routing algorithms have been proposed based on a routing table of size $O(n)$ at each node, where $n$ is the number of nodes in the network.[2] For each destination node the table contains the link to be traversed.

Recently, Santoro and Khatib[1] and van Leeuwen and Tan[3] have proposed a general method, called an interval routing scheme, where routing is provided by compact routing tables of size $O(d)$, where $d$ is the degree of a node. Optimal interval routing algorithms are known for networks with regular topology (such as trees, various types of rings and grids, complete bipartite networks, hypercubes and some combinations of them). For non-regular cases there is an interval routing algorithm optimal for cycle-free networks, and at most within a factor of two from optimality for networks with arbitrary topology.

In this paper we show that the interval routing algorithm cannot be optimal in networks with arbitrary topology. We prove that in the worst case the interval routing algorithm chooses a route which is half as long as the diameter of the network. This answered the question posed by van Leeuwen and Tan.[3]

### 2. Definitions and Notation

A network $G$ is an undirected graph $\langle V, E\rangle$ with a finite set $V$ of vertices and a finite set $E \subseteq V \times V$ of edges. For exclusively technical reasons, each edge in the graph will be replaced by two edges (oriented in opposite directions).

The *interval labelling* of the graph $G = \langle V, E\rangle$ is the labelling of vertices and edges of $G$ satisfying two conditions.

(1) The vertex label is an integer from $\langle 1, |V|\rangle$ such that labels of vertices in $V$ are all supposed to be distinct.

(2) The label of an edge in $E$, directed from a vertex $v$, is

(i) an interval label of the form $\langle i,j\rangle$ for vertex labels $i,j$ (all vertex labels are considered to be cyclically ordered), where all intervals for edges in $G$ directed from the vertex $v$ are pairwise disjoint; or

(ii) a complement label $\delta$, where at most one edge in $G$ among edges directed from the vertex $v$ is assigned by this label; or

(iii) a null label $\lambda$, which means that the edge assigned by this label is insignificant with respect to the routing.

A set of all interval labellings of a graph $G$ is called the *interval labelling scheme* of $G$.

Given a graph $G$, a pair of nodes in $G$ and an interval labelling $\mathscr{L}$ of $G$, one can define an *interval routing algorithm* $\mathscr{A}(\mathscr{L})$ in the following way:

*Algorithm* $\mathscr{A}(\mathscr{L})$

**begin**
  **comment** according to a fixed interval labelling $\mathscr{L}$ a current vertex label is $i$ and a destination vertex label is $j$;
  **if** $i = j$
    **then** the destination vertex has been successfully reached;
  **if** $j$ belongs to an interval label of some edge directed from the vertex with the label $i$
    **then** route to the new current vertex through this edge
    **else** route to the new current vertex through the edge with the complement label
**end**

It is obvious that the interval routing algorithm based on any interval labelling $\mathscr{L}$ for a graph $G$ can fail in finding the route between two vertices of $G$ due to a cycle in the route. An interval labelling $\mathscr{L}$ of a graph is valid (according to the routing algorithm $\mathscr{A}(\mathscr{L})$), if for two arbitrary vertices $u, v$ of $G$ the algorithm $\mathscr{A}(\mathscr{L})$ successfully reaches $v$ from $u$ if and only if there is a path in the graph $G$ between $u$ and $v$.

For a given graph $G$ it is decidable in $O(|V|^2)$ whether an interval labelling of $G$ is valid or not. Furthermore, for every graph $G$ there exists a valid interval labelling of $G$.

We note that the valid interval labelling scheme as defined by van Leeuwen and Tan[3] is a special case of the valid interval labelling defined in this paper, and thus lower-bound results proved in the next section are also valid for their interval routing model.

We need the following notation:

distance $(u, v, G)$ the number of edges in the shortest path between vertices $u$ and $v$ in the graph $G$;

diameter $(G)$ = maximum {distance $(u, v, G)$ | $u, v \in V$};

distance $(\mathscr{A}(\mathscr{L}), u, v, G)$ the number of edges in the path routed by $\mathscr{A}(\mathscr{L})$ between vertices $u$ and $v$ in the graph $G$;

distance $(\mathscr{A}(\mathscr{L}), G)$ = maximum {distance $(\mathscr{A}(\mathscr{L}), u, v, G)$ | $u, v \in V$};

name $(v)$ the label of the vertex $v$.

Finally we introduce two notions which enable us to relate the efficiency of routing algorithms.

An interval routing algorithm $\mathscr{A}(\mathscr{L})$ is *optimal* on $G = \langle V, E\rangle$, if for all $u, v \in V$ it holds

distance $(\mathscr{A}(\mathscr{L}), u, v, G)$ = distance $(u, v, G)$.

An interval routing algorithm $\mathscr{A}(\mathscr{L})$ is $\alpha$-efficient on $G$, $\alpha \geqslant 1$, if it holds

distance $(\mathscr{A}(\mathscr{L}), G) \leqslant \alpha$ diameter $G$).

### 3. Main result

We show that the interval routing algorithm is not optimal, i.e. there is a graph $G$ such that the interval routing algorithm $\mathscr{A}(\mathscr{L})$ is not optimal on $G$ for any valid interval labelling $\mathscr{L}$. In fact we prove much stronger result, namely that the interval routing algorithm cannot be $\frac{3}{2}$-efficient.

**Theorem 1**

*There is a graph $G$ such that for any valid interval labelling $\mathscr{L}$ and for any rational number $\alpha \in \langle 1, \frac{3}{2}\rangle$ the interval routing algorithm $\mathscr{A}(\mathscr{L})$ is not $\alpha$-efficient on $G$.*

*Proof*

We investigate the graph $G_{s,k} = \langle N, E\rangle$ of the size $2ks - s + 2$ and of the diameter $2k$ for $k > 2$ and $s \geqslant 14$, where

$$N = \{v_{s,k} \mid 1 \leqslant i \leqslant s, 1 \leqslant j \leqslant 2k-1\} \cup \{u, w\}$$

and

$$E = \{(u, v_{i,1}) \mid 1 \leqslant i \leqslant s\} \cup \{(w, v_{i,2k-1}) \mid 1 \leqslant i \leqslant s\} \cup \{(v_{i,j}, v_{i,j+1}) \mid 1 \leqslant i \leqslant s, 1 \leqslant j \leqslant 2k-2\}.$$

Every edge in $G_{s,k}$ is considered to be replaced by two edges oriented in opposite directions. We shall consider an arbitrary valid interval labelling $\mathscr{L}$ for the graph $G_{s,k}$. We determine the pair of vertices $w_1$ and $w_2$ in $G_{s,k}$ such that the following property is satisfied:

the interval routing algorithm $\mathscr{A}(\mathscr{L})$ finds a path between vertices $w_1$ and $w_2$ of the length at least $\frac{3}{2}$ diameter $(G_{s,k}) + \frac{1}{2}$. (1)

Consider vertices $v_{i,k}$ for $1 \leqslant i \leqslant s$, $s \geqslant 14$. From the definition of interval labelling it is evident that at least one edge directed from the vertex $v_{i,k}$ must contain an interval label. The graph $G_{s,k}$ is symmetrical according to the 'axis vertices' $v_{i,k}$ for $1 \leqslant i \leqslant s$. Hence, without loss of generality we suppose that there are at least $s/2$ edges of the form $(v_{i,k}, v_{i,k-1})$ containing interval labels. In $\mathscr{L}$ we denote edges as $(v_{i,k}, v_{i,k-1})$ for $1 \leqslant i \leqslant s/2$. Furthermore, at least $s/2 - 2$ pairs of edges $(u, v_{i,1})$ and $(w, v_{i,2k-1})$ for $1 \leqslant i \leqslant s/2$ contain interval labels. In $\mathscr{L}$ we denote these pairs as $(u, v_{i,1})$ and $(w, v_{i,2k-1})$ for $1 \leqslant i \leqslant s/2 - 2$.

The interval label of the edge $(u, v_{i,1})$ for $1 \leqslant i \leqslant s/2 - 2$ must contain names of vertices $v_{i,1}, \ldots, v_{i,k-1}$. If this interval does not contain, let us say, the name of the vertex $v_{i,t}$ for some $t$ satisfying $1 \leqslant t \leqslant k-1$, then the property (1) is fulfilled because the shortest path between vertices $u$ and $v_{i,t}$, not containing the edge $(u, v_{i,1})$, is of the length at least $\frac{3}{2}$ diameter $(G_{s,k}) + \frac{1}{2}$.

If $I_p$ is the smallest interval containing names of all vertices $v_{p,1}, \ldots, v_{p,k-1}$ for $1 \leqslant p \leqslant s/2 - 2$, then in $\mathscr{L}$ the following relation holds:

$$I_i \cap I_j = \varnothing \quad \text{for} \quad 1 \leqslant i \neq j \leqslant s/2 - 2 \qquad (2)$$

If name $(v_{j,t})$ for some $t$ satisfying $1 \leqslant t \leqslant k-1$ belongs to $I_i$ for $i \neq j$, $1 \leqslant i \leqslant s/2 - 2$, then the interval routing algorithm $\mathscr{A}(\mathscr{L})$ finds the path between vertices $u$ and $v_{j,t}$, containing the edge $(u, v_{i,1})$, of the length at least $\frac{3}{2}$diameter$(G_{s,k})+\frac{1}{2}$ and the property (1) is fulfilled.

By analogy, if $Y_p$ is the smallest interval containing names of all vertices $v_{p,k+1},\ldots,v_{p,2k-1}$ for $1 \leqslant p \leqslant s/2-2$, then in $\mathscr{L}$ the following relation holds

$$Y_i \cap Y_j = \varnothing \quad \text{for} \quad 1 \leqslant i \neq j \leqslant s/2-2. \tag{3}$$

If $K_i$ is the interval label of the edge $(v_{i,k}, v_{i,k-1})$ for $1 \leqslant i \leqslant s/2-2$ in $\mathscr{L}$, then $K_i$ must contain names of all vertices $v_{i,1},\ldots,v_{i,k-1}$ and also names of all vertices $v_{1,1},\ldots,v_{\frac{s}{2}-2,1}$. On the other hand, $K_i$ cannot contain names of vertices $v_{i,k+1},\ldots,v_{i,2k-1}$. Let $x$ and $y$ be two vertices with the following property:

$$\text{name}(x) = \text{minimum}\{\text{name}(v_{j,1}) \mid 1 \leqslant j \leqslant s/2-2\}$$

and

$$\text{name}(y) = \text{maximum}\{\text{name}(v_{j,1}) \mid 1 \leqslant j \leqslant s/2-2\}.$$

Without loss of generality, suppose that in $\mathscr{L}$ it holds that $x = v_{\frac{s}{2}-2,1}$ and $y = v_{\frac{s}{2}-2,1}$. For an arbitrary integer $i$ satisfying $1 \leqslant i \leqslant s/2-4$ it follows that $K_i$ contains names of all vertices $v_{j,t}$ for $1 \leqslant j \leqslant s/2-4$, $1 \leqslant t \leqslant k-1$, but that $K_i$ does not contain names of vertices $v_{m,r}$ for $1 \leqslant m \leqslant s/2-4, k+1 \leqslant r \leqslant 2k-1$. Hence, if $I$ denotes the smallest interval containing $I_1 \cup \ldots \cup I_{s/2-4}$, the following equation holds:

$$I \cap (Y_1 \cup \ldots \cup Y_{\frac{s}{2}-4}) = \varnothing \tag{4}$$

Note that $s/2-4 \geqslant 3$. By equation (4) there are two indices $m_1$ and $m_2$ satisfying $1 \leqslant m_1 \neq m_2 \leqslant s/2-4$, such that there does not exist a vertex $x$ from the set $\{v_{i,j} \mid 1 \leqslant i \leqslant s/2-4, 1 \leqslant j \leqslant k-1\}$ whose name is between the name of a vertex from the set $\{v_{m_p,j} \mid k+1 \leqslant j \leqslant 2k-1\}$ and the name of a vertex from the set $\{v_{m_q,j} \mid k+1 \leqslant j \leqslant 2k-1\}$ for $p = 1$, $q = 2$ and $q = 1$, $p = 2$.

In particular consider the following case.

$$\text{name}(v_{1,1}) < \text{name}(v_{m_1,k+1}) < \text{name}(v_{m_2,k+1}) \tag{5}$$

If name$(v_{m_2,k+1})$ belong to the interval $I_{m_2}$, then by the inequality (5) there are two vertices $u$ and $v_{m_1,k+1}$ such that $\mathscr{A}(\mathscr{L})$ finds the path between them of the length at least $\frac{3}{2}$diameter$(G_{s,k})+\frac{1}{2}$ and the property (1) holds. We consider next that name$(v_{m_2,k+2})$ does not belong to the interval $I_{m_2}$.

By equation (2) there are two vertices $u$ and $v_{m_2,k+1}$ such that $\mathscr{A}(\mathscr{L})$ finds the path between them of the length at least $\frac{3}{2}$diameter$(G_{s,k})+\frac{1}{2}$ and the property (1) is satisfied.

Evidently the foregoing argument can apply also in the case dual to (5). This completes the proof of Theorem 1.

**Corollary 1**

*There is a graph $G$ such that for any arbitrary valid interval labelling $\mathscr{L}$ of $G$ the routing algorithm $\mathscr{A}(\mathscr{L})$ satisfies the lower bound*

$$\text{distance}(\mathscr{A}(\mathscr{L}), G) \geqslant \tfrac{3}{2}\text{diameter}(G)+\tfrac{1}{2}.$$

Santoro and Khatib[1] described a valid interval labelling $\mathscr{L}$ for an arbitrary graph $G$ such that the interval routing algorithm $\mathscr{A}(\mathscr{L})$ satisfies the upper bound

$$\text{distance}(\mathscr{A}(\mathscr{L}), G) \leqslant 2\,\text{diameter}(G).$$

We remark that on the graph $G_{s,k}$ the interval routing algorithm is tight with respect to the lower bound given in Theorem 1.

## 4. Final remarks

The question whether there is an optimal interval routing algorithm for any arbitrary network is answered in a negative way. It is shown that for certain networks the interval routing algorithm finds at least half as long a route as the diameter of the network. It means that in the worst case the interval routing algorithms are at least 'half as bad' with respect to the length complexity as the optimal routing algorithms based on the routing tables in implementing the interval routing algorithms in large sparse communications networks.

The technique used in the proof of Theorem 1 can be applied also in proving lower bounds for related problems (i.e. multi-labelling interval routing algorithms are not optimal on certain graphs or interval routing algorithms are not optimal on row-column wrap-around grids).

Another problem concerning interval routing remains unresolved. The question is whether there is an interval routing algorithm reaching the lower bound presented in Theorem 1. Another interesting issue concerns the 'average' complexity analysis of routing in terms of the sum of distances between all pairs of vertices.

P. RUŽIČKA
VUSEI-AR, Dúbravská 3, 842 21 Bratislava, Czechoslovakia

**References**

1. N. Santoro and R. Khatib, Labelling and implicit routing in network. *The Computer Journal* **28** (1), 5–8 (1985).
2. W. D. Tajibnapis, A correctness proof of a topology information maintenance protocol for a distributed computer network. *CACM* **20** (7), 477–485 (1977).
3. J. van Leeuwen and R. B. Tan, Interval routing. *The Computer Journal* **30** (4), 298–307 (1987).



# Announcement

30 MARCH to 1 APRIL 1992

**Eighth International Conference on Software Engineering for Telecommunication Systems and Services,** Florence, Italy

'SETSS 92' is being organised by the Institution of Electrical Engineers (IEE). It will seek to explore critical analyses of current engineering practices and the need for change, together with solutions to these needs, advanced ideas and future trends.

The intended audience will include software engineers engaged in the development, exploitation and maintenance of telecommunication systems and services and those responsible for the appropriate enabling technologies and supporting tools. The conference will also encourage contact with practitioners in related fields of application.

Emphasis will be on technical contributions covering concepts, principles, methods and tools for software and systems engineering relevant to telecommunications.

Conference Services, IEE, Savoy Place, London WC2R 0BL, UK. Tel: 071 240 1871, ext. 222. Fax: 071 240 7735.

8–10 APRIL 1992

**12th ARTS, Advances in Reliability Technology Symposium, UMIST, Manchester**
AEA Technology–SRD, in association with the University of Bradford, UMIST and the University of Liverpool.

**Objective**

This symposium is to discuss, explore and demonstrate the advances which researchers and practitioners have made in their development of reliability engineering. Typical topics include the following.

- Modelling and assessments
- Management and design for reliability
- Availability and economic analysis
- Mechanical and structural reliability
- Common cause failure analysis
- Human factors
- Electronic systems reliability
- Software reliability
- Data collection and analysis
- Practical applications and case studies
- Manufacturing and processing systems
- Artificial intelligence and expert systems

**Registration**

Registration forms and further programme details will be made available a few months before the event and will be sent to those who write to the address given below. Early registration is recommended.

Ms Kay Stanley, AEA Technology–SRD, Wigshaw Lane, Culcheth, Warrington WA3 4NE, UK.

8–13 AUGUST 1993

**5th International Conference on Human–Computer Interaction (HCI International '93)** to be held at the Hilton at Walt Disney World Village, Orlando, Florida.

Further information may be obtained from the conference chair, Dr Gavriel Salvendy, Grissom Hall, Purdue University, West Lafayette, IN 47907, USA. Telephone: 317-494-5426. Fax: 317-494-0874. E-mail: salvendy@ecn.purdue.edu.