# Algorithms for De Bruijn Sequences - A Case Study in the Empirical Analysis of Algorithms 

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#### Abstract

The following paper presents an analysis of algorithms for De Bruijn sequences. It is unusual in that the analysis is carried out empirically rather than analytically.


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## 1. THE ANALYSIS OF ALGORITHMS

Most published work about the analysis of algorithms is analytic because, when analytic results can be obtained, they are usually decisive, often elegant and sometimes beautiful. However, such analyses are not always available either because they are mathematically intractable or because the analysis is indecisive. In such cases, empirical analysis may be used to distinguish the effectiveness of one algorithm from another. Indeed, this situation is more common than the computer science literature might lead you to believe. This note is an example of such an empirical analysis.

## 2. DE BRUIJN SEQUENCES

Definition. Let $S=\{0,1, \ldots, m\}$ be an alphabet of $m+1$ symbols and consider all words of length $n$ from $S$. Let $L=(m+1)^{n}$. An $(m, n)$ de Bruijn sequence $B_{m, n}$ is a sequence

$$
a_{1} a_{2} a_{3}, \ldots, a_{L}
$$

with each $a_{i} \varepsilon S$ such that every word $w$ of length $n$ from $S$ is realized as

$$
\begin{equation*}
a_{i} a_{i+1}, \ldots, a_{i+n-1} \quad(0 \leqslant i \leqslant L) \tag{1}
\end{equation*}
$$

for exactly one $i$, where each subscript in (1) is to be interpreted in modulo $L$.

For example, with $m=1$ and $n=3$ so that $L=8$, the sequence 00010111 contains all binary sequences of length 3 with 110 and 100 being obtained by wraparound from the end of the sequence to the beginning.

The existence of de Bruijn sequences for any $m$ and $n$ can be proved using graph theory or finite field theory [3] but the algorithms suggested by these proofs are clearly very inefficient from both time and space perspectives. The known good algorithms for generating de Bruijn sequences all depend on a strictly combinatorial approach.

## 3. THREE COMBINATORIAL <br> ALGORITHMS FOR DE BRUIJN SEQUENCES

An early combinatorial algorithm due to Martin [2] was inefficient because it required $L$ units of memory. By contrast, the three algorithms to be considered here are all essentially memoryless since each requires $O(n)$ units of storage. Improving the space attributes also improves
the timing properties by requiring much less memory referencing. The three algorithms to be considered here all require $O(L)$ time, clearly the minimum possible.

Two of the three algorithms depend upon the following two definitions:

Definition. Let $S=s_{1}, s_{2}, \ldots, s_{n}$ and let $T=s_{1} s_{2}, \ldots$, $s_{j}, j<n$. Denote by $T^{k}$ the subsequence of $k$ consecutive repetitions of $T$. If $S=T^{k}$ when $k>1$, we say that $S$ is periodic with repetition (or periodicity) $k$ and $T$ is its periodic reduction. If there is no $T$ with $k>1$ for which $S=T^{k}$, we say that $S$ is aperiodic.

Definition. The lexically largest permutation, $\pi_{1}, \pi_{2}, \ldots$, $\pi_{n}$, of an $n$-set $\alpha_{1}, \alpha_{2}, \ldots, \alpha_{n}$, is such that $\pi_{1} \geqslant \pi_{2} \geqslant \ldots \geqslant$ $\pi_{n}$. (Thus, the lexically largest permutation is the numerically largest). If $S=s_{1} s_{2}, \ldots, s_{n}$ and $T=t_{1} t_{2}, \ldots, t_{n}$ are two strings of length $n$, then $S \geqslant T$ iff $s_{i}=t_{i}, i=1, \ldots$, $j$, and $s_{j+1}>t_{j+1}$. Equality holds only when $j=n$. A necklace $S$ of length $n$ is an $n$-sequence with the property that $S>=T$ for every $n$-sequence $T$ which is a cyclic permutation of $S$. Thus, if $S=s_{1}, s_{2}, \ldots, s_{n}$ is a necklace, then

$$
S \geqslant s_{i} s_{i+1}, \ldots, s_{n} s_{1}, \ldots, s_{i-1}
$$

for $2 \leqslant i \leqslant n$.
The first algorithm, due to Fredricksen and Maiorana [1], outputs the lexically largest ( $m, n$ ) de Bruijn sequence by generating successive necklaces of length $n$.

## Algorithm FM

Input: two positive integers $m, n$
Step 1.
Start with the empty string.
Step 2. (Iterative Step) Generate the necklaces of length $n$ whose first symbol is $m$, in decreasing lexicographic order. Append each necklace to the string already generated if it is aperiodic or its periodic reduction otherwise.
Step 3. (Recursive Step) If $m=1$ then append 0 to the string already generated, else append $B_{m-1, n}^{(F M)}$ to the string already generated.
Output: $B_{m, n}^{(F M)}$
Since Algorithm FM requires the generation of successive necklaces, it requires a subalgorithm for this purpose which can be found in [3].

As an example of Algorithm FM, let $n=4$ and $m=$ 2. We obtain:

$$
\begin{aligned}
B_{2,4}^{(F M)}= & 22221222022112210220122002121202111 \\
& 21102101210020201120102001200011110 \\
& 11001010000,
\end{aligned}
$$

where each group shown is a necklace or its periodic reduction.
A second algorithm, Algorithm R, due to Ralston [4], is a variation on Algorithm FM which uses a double recursive approach (see Appendix).

For the third algorithm, due to Xie [5], we need to introduce two more definitions.

Definition. Let $L 1=(m+1)^{n-1}$. If a sequence

$$
0 t_{1} t_{2}, \ldots, t_{L 1-1}
$$

is such that for any $j(1 \leqslant j \leqslant L 1)$, there exists another sequence $j_{1} j_{2}, \ldots, j_{s}(s \leqslant L 1)$ where

$$
j_{s}=0, \quad j_{k} \neq 0 \quad(k<s)
$$

and
$j_{1}=j, \quad j_{k+1}=\left((m+1) * j_{k}+t_{j_{k}}\right) \quad \bmod \quad L 1 \quad(1 \leqslant k<s)$
then $0 t_{1} t_{2}, \ldots, t_{L 1-1}$ is called a label $(m, n)$.
Note: $000, \ldots, 0$ ( $L 10$ 's) is always a label.
As an example, 0001 is a label $(3,2)$ since there are sequences 10 for $j=1,20$ for $j=2$, and 310 for $j=3$. However, 0003 is not a label $(3,2)$ since for $j=3$, the sequence generated by (2) is 3333 .

Definition. A look-up table is a two-dimensional array consisting of $L 1$ rows. Each row consists of an index $i, 0$ $\leqslant i \leqslant L 1-1$, and a sequence of symbols $i_{0}, i_{1}, \ldots, i_{m}$ in which each sequence is a permutation of the set $\{0,1, \ldots$, $m\}$.

Let us now consider the special look-up table shown in Table 1 in which $0 t_{1} t_{2}, \ldots, t_{L 1-1}$ is a label ( $m, n$ ) and $x x$, $\ldots, x$ in row $i$ is $01 \ldots\left(t_{i-1}\right)\left(t_{i+1}\right) \ldots m$.

Table 1

| Index | Sequence |  |  |  |  |
| :--- | :--- | :--- | :--- | :--- | :--- |
| 0 | 0 | 1 | 2 | $\cdots$ | $m$ |
| 1 | $t_{1}$ | $x$ | $x$ | $\cdots$ | $x$ |
| $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ |
| $\dot{L 1-1}$ | $i$ | . | $\cdot$ | $\cdot$ | . |
|  | $L 1-1$ |  | $x$ | $\cdots$ | $x$ |

Xie proves that Algorithm X, associated with Table 1, generates a $(m, n)$ de Bruijn sequence. In particular, if $0 t_{1}$, $t_{2}, \ldots, t_{L 1-1}=00, \ldots, 0$ then the sequence generated is the lexically largest ( $m, n$ ) de Bruijn sequence.

## Algorithm X

Input: positive integers $m, n$
Step 1. Generate a label $(m, n)$ represented by $0 t_{1}, t_{2}, \ldots, t_{L 1-1}$
Step 2.
Construct Table 2 as described above
Step 3. Start with the index $i=0$ and set the sequence to be generated to be empty
Step 4. (Iterative Step)
If the row with index $i$ in the look-up table is empty, stop; otherwise append the rightmost symbol $s$ in this row to the sequence already generated, remove this symbol from the row and evaluate

$$
i \leftarrow((m+1) * i+s) \quad \bmod \quad L 1
$$

Output: an ( $m, n$ ) de Bruijn sequence
For example, with $m=2, n=3, L 1=9$. If we take the label in Step 1 to be all zeros, then the look-up table is:

Table 2

| Index | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| :--- | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| Sequence | 012 | 012 | 012 | 012 | 012 | 012 | 012 | 012 | 012 |

Applying Step 4, we obtain:

$$
B_{2,3}^{(X)}=222122021121020120011101000
$$

This remarkable but rather unintuitive algorithm is, as we shall see in the next section, very efficient because of the extremely simple calculation in Step 4.

## 4. RESULTS

We implemented Algorithms FM, R and X in Pascal on a VAX $11 / 785$ and a Burroughs B7900. Since the results were similar on both computers, we give the results only for the former. The implementation, so far as possible, had similar characteristics with respect to modularity and data structures. The lines of code for the three algorithms were as follows: FM-155, R-318, X-75. The table which follows gives the results for the three algorithms for various values of $m$ and $n$.

Table 3. Time to generate $B_{m, n}$ on a VAX 11/785 (cpu secs)
\(\left.\begin{array}{llllllllll} <br>

\& m \& 2 \& 2 \& 2 \& 2 \& 2 \& 2 \& 3 \& 3\end{array}\right]\)|  |
| :--- |
|  |
| $n$ |

The superiority of Xie's algorithm is clear, particularly so for increasing values of $n$ for each $m$. This superiority is not obvious from the algorithms themselves nor could an analytic analysis have shown this since all three algorithms are $O(L)$ for time. A posteriori we may infer that the simplicity and elegance of the idea in Xie's algorithm accounts for its computational efficiency.

Empirical studies of the performance of algorithms whose analytic orders are the same will not usually result in quite such definitive results as obtained in this case. Still computer scientists and mathematicians should be willing to take an experimental approach to the analysis of algorithms when analytic approaches are not fruitful.

## 5. APPENDIX

Definition. An aperiodic block is a string of aperiodic necklaces of decreasing numerical value with $m$ 's in the same position in each necklace (e.g. with $m=2, n=4$, we get 221122102201 2200).

Definition. Let $T$ be the periodic reduction of a periodic necklace $S$ with repetition $k$ (e.g. $S=2121, T=$ $21, k=2$ ). Let $t$ be the number of ( $m-1$ )s in $T, u=$ $m^{t}-1$ and $h_{0}<h_{1}<h_{2}<\ldots<h_{u}$ be the strings with $m$ s in the same position as in $T$ and nowhere else (e.g. $t=1, u$ $=2^{1}-1=1, h_{0}=20<h_{1}=21$ ). Let $B_{u, k}^{(R)}$ be the de Bruijn sequence generated by Algorithm $\mathbf{R}$ below for $m$ $=u, n=k$ (e.g. for $u=1, k=2$, this turns out to be $B_{1,2}^{(R)}$ $=1100$ ). A periodic block is one in which each symbol $j$ in $B_{u, k}$ is replaced by $h_{j}$ (e.g. 21212020 ).

Definition. A group $G_{j}$ is the sequence of periodic and aperiodic blocks whose initial $j$ symbols are each $m$,
ordered by the value of the initial block of length $n$ in each. For example, with $m=2, n=4, j=1$, then
$\mathrm{G}_{1}=\begin{aligned} & 21212020 \\ & 21112110210121002011201020012000\end{aligned}$
(aperiodic block)
Then here is Algorithm R:
Input: two positive integers $m, n$
Step 1.
Start with the string consisting of the single symbol m
Step 2. (Iterative Step)
For $j=n-1, n-2, \ldots, 1$ append $G_{j}$ to the string already generated. (This will generally involve a recursive call to the algorithm to compute $B_{u, k}^{(R)}$ )
Step 3. (Recursive Step)
If $m=1$, then append 0 to the string already generated; else, append $B_{m-1, n}^{(R)}$ to the string already generated
Output: $B_{m, n}^{(R)}$
For $n=4, m=2$, the following output is obtained (Ralston [3])

$$
\begin{aligned}
& B_{2,4}^{(R)}=2 \\
& 22212220 \\
& 2211221022012200 \\
& 212120202111211021012100201120102001 \\
& 2000 \\
& 111101100101000 \\
& 0
\end{aligned}
$$

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