Letters

Correspondence on *The Millar polyhedron and its use in the construction of octrees*

Bowyer *et al.* [3] criticized Millar *et al.* [8, 11] for coining the name 'Millar Polyhedron' for a 3D Peano curve. In response, Millar *et al.* claimed the name to be a label for an instance of such a 3D curve based on the 2D Hilbert curve, together with their tabular implementation. I agree with Bowyer *et al.* that the name Millar Polyhedron is an unnecessary neologism. The earliest explicit mention I have found of the generalization of Hilbert's curve from 2D to 3D dates to 1912 [13] though it is implicit in work of Peano's dated 1908 [12]. Curves similar to Millar *et al.*'s, all based on the same seed curve M_1 [11, Figure 1], have been described by, for example, Aleksandrov *et al.* [1], Bially [2], Butz [4, 5], Quinqueton and Berthod [14], and Stevens *et al.* [15]. These curves differ slightly from one another and from Millar's because they use different transformations from M_{i-1} to M_i [11, Figure 2]. In addition, Millar *et al.*'s tabular method of construction is not original; tabular methods have been used before, either explicitly [15] or implicitly [2]. Lea [10] has previously used Hilbert curves for maintaining multi-dimensional data.

Different authors have called these curves by different names. Butz [4, 5] explicitly generalizes the name 'Hilbert's curve' to include n-dimensional versions of the original 2D curve. Aleksandrov *et al.* [1] called it a 'multidimensional analog of Hilbert's curve'. Quinqueton and Berthod [14] gave such space-filling curves the general name Peano curves, citing and illustrating Peano's curve and Hilbert's curve as 2D examples. Stevens *et al.* [15] called it an orthogonal three-dimensional Peano curve, citing Hilbert as the originator of the 2D version on which it is based. Bially [2] left his curve unnamed.

I believe Bially [2] and Butz [4] were, in 1969, the first to describe algorithms for generating 3D Hilbert curves of finite order. Bially provided a state diagram (his Figure 3) for constructing his curve which can be generated using only 12 tables (half the number used by Millar *et al.*). It is simple to write down these tables from Bially's state diagram. Butz [4, 5] gave general algorithms for constructing an n-dimensional Hilbert curve of order m. Their algorithms provided both direct and inverse mappings of a point in the n-dimensional space to and from its image on the curve, just as Millar *et al.*'s method did 24 years later. The papers were cited in a review of pattern analysis by Kanal [9, p. 1210]. It is interesting to note that these general algorithms pre-date the many algorithms for the construction of the 2D Hilbert curve; see, e.g., Cole [7] for a review.

Aleksandrov *et al.* [1] described a recursive algorithm for generating the multidimensional coordinates of a point from its position on a Hilbert curve. Their algorithm is general like Butz's [4, 5] whom they cited.

Quinqueton and Berthod [14], in 1981, described a locally adaptive Peano scanning algorithm. Their algorithm, too, is general and can generate any n-dimensional Hilbert curve. The octree construction of Millar *et al.* is equivalent to a 3D application of Quinqueton and Berthod's algorithm. I recommend this paper to readers interested in the early work on the use of space-filling curves.

Stevens *et al.* [15], in 1983, gave two examples of 3D Hilbert curves of order 2. They said "For three dimensions, a table of 24 curves is required," which is exactly the number of tables (curves) that Millar *et al.* use. I consider it safe to conclude that Stevens *et al.* used a tabular method equivalent to Millar *et al.*'s.

Cole [6], in 1986, briefly discussed the generalization of his tabular method of constructing 2D Hilbert curves to the construction of n-dimensional Hilbert curves. (Cole used the term Hilbert polygon.)

I believe these references irrefutably show that the name Millar Polyhedron is an unnecessary neologism as it denotes a particular curve which has been recognized as a 3D extension of Hilbert's 2D curve since no later than 1912. The use of a tabular method of constructing such curves has been published previously [2], though that method required fewer tables than Millar *et al.*'s and gave the tables implicitly as a state diagram. It is highly likely that the tabular method of Millar *et al.* has also been used before [15], though the tables were not then published.

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Note

Editor replies: The correspondence on this paper is now closed.