6. Conclusions

A program for the Mercury computer, using the scheme described in this paper, has been tested for a matrix of order 96. The fraction of the total time spent on drum transfers is quite small in this case. For a computer with a fast arithmetic unit in which the backing store was magnetic tape, so that the speed of transfer from the backing store was the effective controlling factor, the speed factor gained by minimizing the number of scans through the matrix would be even greater than that of seven obtained with Mercury.

It is worth noting that in this single respect the Givens' process is better than that of Householder which needs two scans to produce each row of zeros. In other ways the Householder process still has the advantage, requiring only about $\frac{2}{3}n^3$ multiplications as against $\frac{4}{3}n^3$, and $\frac{1}{2}n(n+1)$ backing stores as against n^2. The requirement for n^2 stores can be reduced to $\frac{1}{2}n(n+1)$ if the smaller of $\cos\theta_{ij}$ and $\sin\theta_{ij}$ is stored with an identifying flag, instead of both $\cos\theta_{ij}$ and $\sin\theta_{ij}$. There are consequent losses both in speed and accuracy. If the eigenvectors are not required then there is no need to store the cosines and sines, and the Givens' process may become quite attractive when a tape backing store is being used. The technique described in this paper may well have counterparts in other transformations of a similar nature.

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Correspondence

To the Editor,
The Computer Journal.

Sir,

A few comments on the significance of Allen's useful approximation (Allen, 1959) may be helpful to other readers. Given an empirical function $f(x, y)$, the problem is to fit it numerically by finding functions $a_r(x)$ and $b_r(y)$ such that

$$f(x, y) \simeq \sum_{r=1}^{n} a_r(x) b_r(y).$$

If the variables are defined over a rectangular lattice, we may write this in matrix notation $F \simeq AB'$ where A and B each have n columns. The mathematical interest of the problem is that a solution can be found, as Allen shows, in terms of the eigenvectors of FF' such that $A'A$ and $B'B$ are simultaneously diagonal. This gives the hint that a *least squares fit* of F to AB' has been obtained, though this is nowhere stated in the paper.

Allen fits FF' to AA', and the significance of this is obscure. If, instead, we look for a least squares fit of F to AB' from the start, the eigenvalue equation is very simply obtained. Thus, differentiating the sum of squares of the elements of $F - AB'$ with respect to the elements of A and B gives

$$(F - AB')B = 0 \quad (1) \qquad (F' - BA')A = 0 \quad (2)$$

from which we immediately obtain the eigenvalue equation

$$FF'A = A(B'B)(A'A).$$

By choosing $B'B$ to be diagonal without assuming it to be normalized as a unit matrix, we may write the admissible solutions in the form

$$A = VT \quad (3) \qquad B = F'A \quad (4)$$

where V is the unitary matrix of eigenvectors of FF' and the matrix

$$T = \begin{bmatrix} I_n \\ 0 \end{bmatrix}$$

selects n of these for inclusion in A. In this form of solution, we are not troubled by factors $\sqrt{\lambda}$ because they have been absorbed by F' in equation (4) above.

There is a further result of importance to the numerical analyst, pointed out to me independently by Mr. E. D. Farmer and Dr. D. P. Jenkins. By considering the trace of $(F - AB')(F' - BA')$, and denoting the complete diagonal eigenvalue matrix by Λ, it is not difficult to show that the *sum of the squared errors* in the fit is given by

$$Tr(\Lambda - TT'\Lambda),$$

which is simply the *sum of the omitted eigenvalues.*

Malvern. Yours faithfully,
17 April 1961. P. M. Woodward.

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