Newton–Cotes type quadrature formulas with terminal corrections

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A number of quadrature formulas for equidistant argument are proposed which combine optimized weighting of the ordinates with the use of a limited number of derivatives, central differences or differences of first derivatives at the end of the range of integration. The comparative errors of the new formulas, as well as of the standard formulas and those given by Salzer, are discussed qualitatively and illustrated on a number of examples.

Nature of the approximation formulas

Numerical quadrature formulas for finite ranges of integration and equidistant arguments are mostly based on the trapezoidal rule in which the integral

$$I = \int_a^b f(x)dx \tag{1}$$

is replaced in zero order by the sum*

$$I_{100}(a, h, r) = h[\tfrac{1}{2}f(a) + \tfrac{1}{2}f(b) + \sum_{k=1}^{r-1} f(a + kh)], \quad b = a + rh. \tag{2}$$

To this expression correction terms can be added, which may be of several types (*cf.* e.g. Lanczos 1957, Kopal 1961); of the standard quadrature formulas the following are the most relevant for the purposes of the present paper.

(1) The Euler–Maclaurin formula, in which the correction is expanded as a series in the odd terminal derivatives (T.D.'s):

$$I_{1M0} = I_{100} - \sum_{m=1}^{M}{}' \frac{B_{m+1}}{(m+1)!} h^{m+1} \left[\frac{d^m f(b)}{db^m} - \frac{d^m f(a)}{da^m}\right] \tag{3}$$

where the prime on the summation sign indicates that the sum is to be taken over odd values of the index only and the B_{m+1} are the (signed) Bernoulli numbers

$$B_2 = \frac{1}{6}, B_4 = \frac{-1}{30}, B_6 = \frac{1}{42}, B_8 = \frac{-1}{30}, \ldots \tag{4}$$

(2) Gauss's modification of Gregory's formula, which uses terminal central differences (T.$\mu\delta$'s) of odd order

$$I_{10N} = I_{100} - h \sum_{m=1}^{N}{}' \frac{C_{n+1}}{(n+1)!}[\mu\delta_h^n f(b) - \mu\delta_h^n f(a)] \tag{5}$$

where

$$\mu\delta_h g(x) = \tfrac{1}{2}[g(x+h) - g(x-h)]; \quad \delta_h^2 g(x) = g(x+h) - 2g(x) + g(x-h), \tag{6}$$

and correspondingly for higher powers in δ_h^2. The coefficients C_{n+1} can be calculated by integrating the Newton–Bessel interpolation formula:

$$C_2 = \frac{1}{6}, C_4 = -\frac{11}{30}, C_6 = \frac{191}{84}, C_8 = -\frac{2{,}497}{90}, \ldots \tag{7}$$

* The meaning of the triple indices of I will be explained in equation (9) below.

(3) The Newton–Cotes formulas, in which the total range of integration is divided into a number of (preferably equal) subranges, and within each subrange the function is essentially approximated by a Lagrangian interpolation polynomial, which is integrated. This results in summations in which the tabular values of f enter with unequal coefficients, in contrast to equation (2).

$$I_{L00}(a, h, L) = h \sum_{l=0}^{L} a_l f(a + lh). \tag{8}$$

These formulas can also be considered as the trapezoidal rule (2) with corrections involving *even* differences up to the highest order (L or $L - 1$) which can be formed *within* each subrange.

All the formulas quoted above suffer from limitations if a high order of accuracy is required. The Euler–Maclaurin formula may necessitate the calculation of high derivatives, which in most cases is increasingly difficult; the central difference method requires knowledge of the function beyond its range of integration; for the most convenient use of the higher Newton–Cotes formulas the total number of intervals r should be divisible by the order of the subrange L; in addition there is an increasingly marked alternation in the magnitude of the coefficients, leading to alternations in sign in extreme cases, which seems somewhat unnatural from a logical point of view and increases rounding-off errors.

In view of the fact that for many functions the first derivative, though not the higher ones, can be calculated easily, the writer was led to investigate the possibility of combining any two or all three methods, i.e. odd T.D.'s, odd $T.\mu\delta$'s and adjustable coefficients for the tabular values of the integrand. Such expressions, for a basic range of L intervals, would be of the form:

$$I_{LMN}(a, h, L) = h \sum_{l=0}^{L} a_l f(a + lh) + \sum_{m=1}^{M}{}' b_m h^{m+1} \left[\frac{d^m f}{dx^m}\right]_a^{a+Lh} + h \sum_{n=1}^{N}{}' c_n[\mu\delta_h^n f(a + Lh) - \mu\delta_h^n f(a)] \tag{9}$$

where $$a_l = a_{L-l} \tag{10}$$

for reasons of symmetry. The last sum in (9) can be written in the alternative form

$$h \sum_{n=1}^{(N+1)/2} c_n'[\mu\delta_{nh}f(a+Lh) - \mu\delta_{nh}f(a)]. \qquad (11)$$

Formulas of the type (9) yield accurate integrals for all polynomials up to degree P

$$P = L_o + M_e + N_e \qquad (12)$$

where

$$\begin{aligned} L_o &= L + 1\ (L \text{ even}),\ L_o = L \qquad (L \text{ odd}),\\ M_e &= 0 \qquad (M = 0),\ M_e = M + 1\ (M \text{ odd} > 0),\\ N_e &= 0 \qquad (N = 0),\ N_e = N + 1\ (N \text{ odd} > 0). \end{aligned} \qquad (13)$$

The coefficients a_l, b_m and c_n or c_n' in (9) and (11) can easily be determined as the solutions for the unknowns in the $\frac{1}{2}(P+1)$ linear equations obtained by substituting $x^p(p = 0, 2, \ldots, P-1)$ for the integrand, with $h = 1$ and a suitable choice of a, though some of the formulas can be obtained more simply as shown in the Appendix. The resulting coefficients are listed in Table 1 for a number of formulas in order of increasing L; those for the pure Newton–Cotes formulas are included in the table, but not for the pure Euler–Maclaurin and central difference methods, which are given by (3) and (4), and (5) and (7). The list could be indefinitely extended, but in the writer's opinion higher order expressions would be of little advantage as they would suffer progressively from the drawbacks enumerated above for the standard formulas. It is usually preferable to subdivide the total range into equal subranges and apply a lower order formula repeatedly; in this case the derivatives and central differences will cancel except at the limits of the total range, and the tabular values at the junction of two subranges appear with coefficient $2a_0$.

Of all the formulas specified by Table 1 those with $L = 2$ or 3 are by far the most useful; they combine high accuracy with reasonable simplicity and small variation in the size of the coefficients of f (apart from the factor $\frac{1}{2}$ for a_0 and a_L). They can be applied particularly simply if we define for even r:

$$\left.\begin{aligned} S_e &= \tfrac{1}{2}f(a) + f(a+2h) + \ldots f(b-2h) + \tfrac{1}{2}f(b),\\ S_o &= f(a+h) + f(a+3h)\\ &\qquad + f(a+5h) + \ldots + f(b-h); \end{aligned}\right\} \qquad (14)$$

and for r divisible by 3:

$$\left.\begin{aligned} S_3 &= \tfrac{1}{2}f(a) + f(a+3h) + \cdots + f(b-3h) + \tfrac{1}{2}f(b),\\ S_{12} &= f(a+h) + f(a+2h) + f(a+4h) + \cdots + f(b-h). \end{aligned}\right\} \qquad (15)$$

Then we can put

$$I_{210} = h(S_o + S_e) + \frac{1}{15}h(S_o - S_e) + \frac{1}{15}h^2[f'(a) - f'(b)], \qquad (16)$$

$$I_{201} = h(S_o + S_e) + \frac{11}{15}h(S_o - S_e) + \frac{1}{90}h[f(a+h) - (a-h) + f(b-h) - f(b+h)], \qquad (17)$$

$$I_{310} = h(S_3 + S_{12}) + \frac{1}{80}h(S_{12} - 2S_3) + \frac{3}{40}h^2[f'(a) - f'(b)], \qquad (18)$$

$$I_{301} = h(S_3 + S_{12}) + \frac{11}{160}h(S_{12} - 2S_3) + \frac{3}{160}h[f(a+h) - f(a-h) + f(b-h) - f(b+h)]. \qquad (19)$$

Comparison with other mixed formulas

Several of the formulas described by the equations (9) and (11), or expressions equivalent to them, have been derived by previous authors. If we consider only one basic subrange, the first and last sums in (9) can be combined to yield

$$I_{LMN} = h\Sigma a_l' f(a + lh) + \Sigma_m' \ldots \qquad (20)$$

where the first sum is now to be taken from $l = -\frac{1}{2}N_e$ to $l = L + \frac{1}{2}N_e$, and

$$a_l' = a_l - \tfrac{1}{2}\,\mathrm{sgn}\, l\, c_{|l|}' - \tfrac{1}{2}\,\mathrm{sgn}\,(L - l)\, c_{|L-l|}'. \qquad (21)$$

In this formulation all the relevant ordinates occur once only, and not twice as in (9) or (11). Coefficients for the formulas (20) when $M = 0$ have been given by a number of writers; the most comprehensive list has been given by Miller (1960) for $N' = \frac{1}{2}N_e = 1, 2$ and -1; the last case, in which the terminal ordinates are not used at all, is not strictly contained in the formula (9).

The expression I_{210}, under the name "corrected Simpson's rule" has been discussed in detail by Lanczos (1957), though the formula (6.17.2) in his book contains an error in sign.

Quadrature formulas making use of first derivatives have been derived by Salzer (1955, 1960) by integrating Hermite osculatory interpolation polynomials over ranges equal to, or smaller than, the range of data utilized. They are of the form

$$I = h\Sigma a_l f(a + lh) + h^2\Sigma b_l f'(a + lh) \qquad (22)$$

and for $L + 1$ data points are accurate for polynomials up to degree $P = 2L + 1$; they should therefore be more accurate than most of the formulas with the same value of L derived in this paper. Against this must be held that the derivatives must be known or calculated at internal points, and that the corresponding terms do not cancel when the integration is carried out over several contiguous subranges.

Formulas similar to Salzer's can be derived which involve first derivatives at other points besides the integration limits, but are not necessarily based on a com-

Table 1

Coefficients occurring in the expressions I_{LMN}. [Cf. equations (9) and (11)]
For $L > 3$, the a_l are given only up to $l = [\frac{1}{2}L]$

$L\ M\ N\ P$	a_0	a_1	a_2	a_3	b_1	b_3	c_1	c_1'	$c_3 = c_2'$
1 1 1 5	$\frac{1}{2}$	$\frac{1}{2}$			$-\frac{11}{120}$		$\frac{1}{120}$	$\frac{1}{120}$	
1 3 1 7	$\frac{1}{2}$	$\frac{1}{2}$			$-\frac{5}{63}$	$\frac{31}{15120}$	$-\frac{1}{252}$	$-\frac{1}{252}$	
1 1 3 7	$\frac{1}{2}$	$\frac{1}{2}$			$-\frac{191}{2016}$		$\frac{23}{2016}$	$\frac{47}{3780}$	$\frac{-31}{60480}$
2 0 0 3	$\frac{1}{3}$	$\frac{4}{3}$	$\frac{1}{3}$						
2 1 0 5	$\frac{7}{15}$	$\frac{16}{15}$	$\frac{7}{15}$		$-\frac{1}{15}$				
2 0 1 5	$\frac{17}{45}$	$\frac{56}{45}$	$\frac{17}{45}$				$-\frac{1}{45}$	$-\frac{1}{45}$	
2 3 0 7	$\frac{31}{63}$	$\frac{64}{63}$	$\frac{31}{63}$		$-\frac{5}{63}$	$\frac{1}{945}$			
2 1 1 7	$\frac{457}{945}$	$\frac{976}{945}$	$\frac{457}{945}$		$-\frac{5}{63}$		$\frac{4}{945}$	$\frac{4}{945}$	
2 0 3 7	$\frac{377}{945}$	$\frac{1136}{945}$	$\frac{377}{945}$				$\frac{-31}{945}$	$\frac{-4}{105}$	$\frac{1}{378}$
3 0 0 3	$\frac{3}{8}$	$\frac{9}{8}$	$\frac{9}{8}$	$\frac{3}{8}$					
3 1 0 5	$\frac{39}{80}$	$\frac{81}{80}$	$\frac{81}{80}$	$\frac{39}{80}$	$-\frac{3}{40}$				
3 0 1 5	$\frac{69}{160}$	$\frac{171}{160}$	$\frac{171}{160}$	$\frac{69}{160}$			$\frac{-3}{80}$	$\frac{-3}{80}$	
3 3 0 7	$\frac{363}{728}$	$\frac{729}{728}$	$\frac{729}{728}$	$\frac{363}{728}$	$-\frac{60}{728}$	$\frac{9}{7280}$			
3 1 1 7	$\frac{4449}{8960}$	$\frac{8991}{8960}$	$\frac{8991}{8960}$	$\frac{4449}{8960}$	$\frac{-39}{448}$		$\frac{27}{4480}$	$\frac{27}{4480}$	
3 0 3 7	$\frac{2049}{4480}$	$\frac{4671}{4480}$	$\frac{4671}{4480}$	$\frac{2049}{4480}$			$\frac{-123}{2240}$	$\frac{-149}{2240}$	$\frac{13}{2240}$
4 0 0 5	$\frac{14}{45}$	$\frac{64}{45}$	$\frac{24}{45}$						
4 1 0 7	$\frac{62}{135}$	$\frac{1024}{945}$	$\frac{32}{35}$		$\frac{-4}{63}$				
4 0 1 7	$\frac{342}{945}$	$\frac{1216}{945}$	$\frac{664}{945}$				$\frac{-16}{945}$	$\frac{-16}{945}$	
4 3 0 9	$\frac{7874}{16065}$	$\frac{16384}{16065}$	$\frac{15744}{16065}$		$\frac{-4}{51}$	$\frac{16}{16065}$			
4 1 1 9	$\frac{34022}{70875}$	$\frac{8192}{7875}$	$\frac{544}{567}$		$\frac{-52}{675}$		$\frac{256}{70875}$	$\frac{256}{70875}$	

Table 1 (*continued*)

L M N P	a_0	a_1	a_2	a_3	b_1	b_3	c_1	c_1'	$c_3 = c_2'$
5 0 0 5	$\frac{95}{288}$	$\frac{125}{96}$	$\frac{125}{144}$						
5 1 0 7	$\frac{22835}{48384}$	$\frac{1875}{1792}$	$\frac{11875}{12096}$		$\frac{-275}{4032}$				
5 0 1 7	$\frac{9355}{24192}$	$\frac{14375}{12096}$	$\frac{22375}{24192}$				$\frac{-275}{12096}$	$\frac{-275}{12096}$	
6 0 0 7	$\frac{41}{140}$	$\frac{54}{35}$	$\frac{27}{140}$	$\frac{68}{35}$					
6 1 0 9	$\frac{3149}{7000}$	$\frac{972}{875}$	$\frac{243}{280}$	$\frac{8}{7}$	$\frac{-3}{50}$				
6 0 1 9	$\frac{241}{700}$	$\frac{1899}{1400}$	$\frac{387}{700}$	$\frac{209}{140}$			$\frac{-9}{700}$	$\frac{-9}{700}$	

Table 2

Coefficients occurring in the expressions I'_{LMN}. [Cf. equations (23) and (24)]

For $L = 1$, $a_0 = a_1 = \frac{1}{2}$. For $L > 2$, the a_l are given only up to $l = 2$

L M N P	b_1'	b_1''	b_3'	b_2''	$b_5' = b_3''$	c_1	c_1	$c_3 = c_2'$
1 3 0 5	$-\frac{1}{12}$	$-\frac{31}{360}$	$\frac{1}{720}$	$\frac{1}{720}$				
1 3 1 7	$-\frac{37}{336}$	$-\frac{131}{1260}$	$\frac{-31}{10080}$	$\frac{-31}{10080}$		$\frac{3}{112}$	$\frac{3}{112}$	
1 3 3 9	$-\frac{1229}{12096}$	$-\frac{2533}{25920}$	$\frac{-176}{90720}$	$\frac{-176}{90720}$		$\frac{221}{12096}$	$\frac{803}{45630}$	$\frac{103}{362880}$
1 5 0 7	$-\frac{1}{12}$	$-\frac{877}{10080}$	$\frac{1}{720}$	$\frac{1}{504}$	$\frac{-1}{6720}$			
1 5 1 9	$-\frac{279}{2400}$	$-\frac{18119}{168000}$	$\frac{-59}{14400}$	$\frac{-6401}{1512000}$	$\frac{103}{3024000}$	$\frac{79}{2400}$	$\frac{79}{2400}$	
1 5 3 11	$\frac{-9959}{76032}$	$\frac{-3413}{29568}$	$\frac{-1579}{190080}$	$\frac{-5057}{665280}$	$\frac{-313}{1774080}$	$\frac{3623}{76032}$	$\frac{6293}{142560}$	$\frac{4001}{2280960}$

L M N P	a_0	a_1	a_2	b_1'	b_1''	$b_3' = b_2''$	c_1
2 3 0 7	$\frac{37}{77}$	$\frac{80}{77}$	$\frac{37}{77}$	$\frac{-17}{231}$	$\frac{-259}{3465}$	$\frac{2}{3465}$	
2 3 1 9	$\frac{4201}{8505}$	$\frac{8608}{8505}$	$\frac{4201}{8505}$	$\frac{-8}{81}$	$\frac{-269}{2835}$	$\frac{-11}{5670}$	$\frac{157}{8505}$
3 3 0 7	$\frac{111}{224}$	$\frac{225}{224}$	$\frac{225}{224}$	$\frac{-9}{112}$	$\frac{-23}{280}$	$\frac{1}{1120}$	
3 3 1 9	$\frac{85017}{170240}$	$\frac{170343}{170240}$	$\frac{170343}{170240}$	$\frac{-129}{1216}$	$\frac{-4299}{42560}$	$\frac{-27}{10640}$	$\frac{1971}{85120}$
4 3 0 9	$\frac{387818}{813645}$	$\frac{851968}{813645}$	$\frac{775008}{813645}$	$\frac{-556}{7749}$	$\frac{-6572}{90405}$	$\frac{128}{271215}$	
4 3 1 11	$\frac{2841698}{5769225}$	$\frac{1951744}{1923075}$	$\frac{103328}{104895}$	$\frac{-5308}{54945}$	$\frac{-5212}{54945}$	$\frac{-32}{18315}$	$\frac{97984}{5769225}$

plete osculatory interpolation. For this purpose the expansion (9) has to be modified by replacing the sum over the odd derivatives by a sum over central differences of the first derivative

$$I'_{LMN} = h \sum_i \ldots + h^2 \sum_{m=1}^{M}{}' \, b'_m \delta_h^{m-1} [f'(a + Lh) - f(a)] + h \sum_n \ldots \quad (23)$$

where the first and last sums are identical with those given in (9) or (11). The sum over m can be written in several alternative forms of which the following is the most useful

$$h^2 \sum b''_m [f'(a + (L + m)h) - f'(a + mh)], b''_{-m} = b''_m \quad (24)$$

with m running from $-(\frac{1}{2}M_e - 1)$ to $(\frac{1}{2}M_e - 1)$. For $M = 1$ the expressions I'_{L1N} are identical with I_{L1N}. For formulas with $M > 1$ the coefficients are tabulated in Table 2. The expressions I_{210}, I'_{131}, I'_{231}, I'_{153} and I'_{253} are equivalent to some of Salzer's formulas, but in the forms (23) or (24) they have the advantage that the correction terms vanish for adjoining subranges, except at the end of the total range.

The general condition for terminal corrections to cancel on juxtaposition of equal subranges is that they are centred on the integration limits and can be expressed as odd-order derivatives or central differences.* Hence they must not include even derivatives such as the one-strip formulas given by Lanczos (1957) in section (6.19) of his book, nor forward and backward differences as in Gregory's formula. The latter, for any finite order of differences, is equivalent to a formula of the type (20) with $M = 0$, though not necessarily with optimized coefficients; hence any refinement thereon by means of better weighting factors or derivatives will only lead back to another formula of the type (20). Nor is it convenient to retain part of these non-central differences as terminal corrections, since for a given end point two corresponding expansions in Δ and ∇ up to order N will differ by quantities of order $N + 1$, and on juxtaposition will only partially cancel. Formulas outside the scope of (9) and (23) which satisfy the cancellation conditions might involve both D^3 and $\delta^2 D$, or mid-interval ordinates; but such cases will not be further investigated here.

Discussion of errors

The error in all the expressions (9) and (23) for a single basic range is of the form

$$I - I_{LMN} = Lh^{P+2} E_{LMN} f^{(P+1)}(\xi)/(P + 1)! \quad (25)$$

where the E_{LMN} are constants and ξ is a value of x lying between $a - \frac{1}{2}N_e h$ and $a + (L + \frac{1}{2}N_e)h$. This form has been chosen so that for unit intervals LE represents the error in the integral of the function x^{P+1} over the range; in particular for the Euler–Maclaurin formula and the central-difference formula the coefficients E_{1M0} and E_{10N} become $-B_{M+2}$ and $-C_{N+2}$ of (4) and (7), respectively. The coefficients E are tabulated in Table 3 in order of increasing P for most of the formulas given in this paper, including (3) and (5) and Salzer's 4-interval formula

$$\int_0^4 f(x)dx = \frac{3,202}{8,505}[f(0) - f(4)] + \frac{8,192}{8,505}[f(1) + f(3)] + \frac{416}{315}f(2) - \frac{116}{2,835}[f'(4) - f'(0)] + \frac{512}{2,835}[f'(3) - f'(1)]. \quad (26)$$

The general trend of the error can be explained by the following argument.

For a prescribed number of data, but adjustable positions, the optimal grouping corresponds to Gauss's quadrature formula, in which the data points are fairly widely spaced in the middle of the range, but clustered towards its ends. Hence a quadrature accurate to order P will have a smaller error the more closely the corresponding interpolation polynomial describes the integrand near the limits. With the formulas discussed in this paper high order T.D.'s will, in general, satisfy this requirement best; first order T.$\mu\delta$.'s and second differences of first derivatives, in turn, will be more advantageous than large values of L, for which the function is equally closely approximated throughout the range, though higher differences are less useful as they involve data too far outside the integration range. As an illustration the results obtained with various formulas having $P = 5$, and with Simpson's rule, for the integral

$$I = \int_{-2}^{2} x \, dx = 256/7; \quad h = 1$$

are listed in order of accuracy in the second column of Table 4; approximations with $P = 7$ and $P = 9$ give the result rigorously.

If the integrand possesses singularities near the range of integration, the above arguments need modification as the higher order derivatives will vary considerably between a and $a + Lh$, and the advantage of a small value of E may be more than offset by a less favourable position of ξ. If we expand the integrand as a Taylor series about the midpoint of a subrange, terms of very high order may contribute appreciably to the integrals. Integrals of the form

$$\int_a^{a+Lh} [x - (a + \tfrac{1}{2}Lh)]^s dx/(\tfrac{1}{2}Lh)^{s+1} \quad (27)$$

form a harmonic sequence, whereas the corresponding expressions from the quadrature formulas are sums of geometric progressions which converge if only internal and terminal points are used, but diverge for formulas employing external points or T.$\mu\delta$.'s (cf. (20)). Terms due to T.D.'s form arithmetic progressions; their rate of divergence increases with M. (It is known that both the Euler–Maclaurin and the central-difference formulas

* This is true only if the symmetry condition (10) holds.

Table 3

Values of the error coefficients E_{LMN}

Primed values of M refer to I'_{LMN}

P	L M N	E	P	L M N	E	P	L M N	E
3	1 1 0	+0·0333	7	1 5 0	+0·0333	9	1 7 0	−0·0758
	1 0 1	+0·367		1 5′ 0	+0·878		1 0 7	−560
	2 0 0	−0·133		1 3 1	+0·0652		2 3′ 1	+1·03
	3 0 0	−0·300		1 3′ 1	+0·163		4 3 0	−3·48
				1 1 3	+0·458		4 1 1	−9·31
5	1 3 0	−0·0238		1 0 5	+27·8		6 1 0	+74·6
	1 3′ 0	−0·107		2 3 0	−0·102		6 0 1	+373
	1 1 1	−0·0738		2 3′ 0	−0·362		(26)	+3·69
	1 0 3	−2·27		2 1 1	−0·279			
	2 1 0	+0·0762		2 0 3	−4·09			
	2 0 1	+0·476		3 3 0	−0·267			
	3 1 0	+0·193		3 3′ 0	−1·04			
	3 0 1	+1·39		3 1 1	−0·772			
	4 0 0	−1·52		3 0 3	−14·7			
	5 0 0	−3·27		4 1 0	+1·63			
				4 0 1	+9·24			
				5 1 0	+3·96			
				5 0 1	+23·6			
				6 0 0	−43·2			

diverge asymptotically if the integrand possesses singularities for any finite complex argument; the same would apply if formulas analogous to those derived in this paper with increasing M or N were employed.) The Newton–Cotes formulas thus appear favoured over expressions with non-zero M and N. On the other hand, the terms (27) enter into the Taylor expansion with coefficients roughly proportional to $(Lh/z)^s$, where z is the modulus of the difference of the midpoint of the range and the nearest singularity, and for high values of P it may be preferable to keep L small by increasing M or N, in spite of the ultimate divergence of the corresponding expressions. Thus the numerical results for the integral.

$$I = \int_1^2 dx/x = \ln 2,\ h = \tfrac{1}{4}$$

are compared in the third column of Table 4. We see that, although the proximity of the singularity at $x = 0$ makes the use of differences particularly dangerous, the length of the basic range still makes the 4-strip Newton–Cotes formula the least accurate of all the expressions with $P = 5$; the same applies to I_{410} if $P = 7$.

Short basic ranges are of special importance if the integration takes place between two complex singularities. This is strikingly illustrated in the last column of Table 4 for the integral

$$I = \int_0^1 dx/(1 + x^2) = \tfrac{1}{4}\pi, \quad h = \tfrac{1}{4}$$

Table 4

Comparison of numerical results

$f(x)$		x^6	x^{-1}	$(x^2 + 1)^{-1}$
a		−2	1	0
b		2	2	1
h		1	0·25	0·25
r		4	4	4
I	P	36·57	0·693 14718	0·785 3981634
I_{130}	5	36·67	1481	39828
I_{210}	5	36·27	1448	39706
I_{111}	5	36·87	1502	39854
I_{201}	5	34·67	1305	3931
I_{400}	5	42·67	1746	5294
I_{200}	3	45·33	2539	3921
I_{131}	7	36·57	14706	3981635
I_{230}	7	36·57	14731	39799
I_{211}	7	36·57	14757	39781
I_{410}	7	36·57	14626	40336
(26)	9	36·57	14712	398124
I'_{130}	5	37·00	15176	39871
I'_{131}	7	36·57	14680	3981641
I'_{230}	7	36·57	14771	39775
I'_{231}	9	36·57	14690	39802

where the singularities occur at $\pm i$. The 4-interval Newton–Cotes rule gives an error several magnitudes greater than the Euler–Maclaurin formula using third derivatives, and is even considerably worse than Simpson's rule. The surprisingly good result obtained with the latter (I_{200}) is to some extent fortuitous since the third derivative vanishes at both limits, and hence the mean value of the fourth derivative vanishes. For the same reason $I_{110} = I_{130}$ in this case.

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Appendix

Simplified derivations of some of the rules

Formulas of the type I_{LM0} with $L = 2$, 3 or 4 are most easily derived by elimination of higher derivatives in the Euler–Maclaurin formula (3) and (4). Thus elimination of the third derivatives from the expressions

$$\int_a^{a+2} f(x)dx = \frac{1}{2}(f_0 + f_2) + f_1 - \frac{1}{12}(f_2' - f_0') + \frac{1}{720}(f_2''' - f_0''') \quad \text{(A1)}$$

and

$$\int_a^{a+2} f(x)dx = f_0 + f_2 - \frac{4}{12}(f_2' - f_0') + \frac{16}{720}(f_2''' - f_0''') \quad \text{(A2)}$$

where f_l is written for $f(a + l)$, leads to

$$\int_a^{a+2} f(x)dx = \frac{7}{15}(f_0 + f_2) + \frac{16}{15}f_1 - \frac{1}{15}(f_2' - f_0') \quad \text{(A3)}$$

which is the formula I_{210} for $h = 1$; it is accurate for polynomials up to fifth degree. Similarly elimination of terms in $d^{M+2}f/dx^{M+2}$ from $I_{1,\,M+2,\,0}$ with intervals h and $2h$ leads to the general formula

$$I_{2M0} = \frac{h}{2^{M+3} - 1}\{(2^{M+2} - 1)(f_0 + f_2) + 2^{M+3}f_1\} - \sum_{m=1}^{M}{}' \frac{h^{m+1}B_{m+1}}{(m+1)!}\,\frac{2^{M+3} - 2^{m+1}}{2^{M+3} - 1}(f_2^{(m)} - f_0^{(m)}) \quad \text{(A4)}$$

and if the intervals are h and $3h$ to

$$I_{3M0} = \frac{h}{3^{M+3} - 1}\left\{\frac{3^{M+3} - 3}{2}(f_0 + f_3) + 3^{M+3}(f_1 + f_2)\right\} - \sum_{m=1}^{M}{}' \frac{h^{m+1}B_{m+1}}{(m+1)!}\,\frac{3^{M+3} - 3^{m+1}}{3^{M+3} - 1}(f_3^{(m)} - f_0^{(m)}). \quad \text{(A5)}$$

A second elimination of $f^{(M+2)}$ from $I_{2,\,M+2,\,0}$ yields

$$I_{4M0} = \frac{h}{(2^{M+5} - 1)(2^{M+3} - 1)}\{(2^{M+2} - 1)(2^{M+5} - 2) \times (f_0 + f_4) + 2^{2M+8}(f_1 + f_3) + (2^{2M+8} - 5\cdot 2^{M+4})f_2 - \sum_{m=1}^{M}{}' h^{m+1}B_{m+1}(2^{M+5} - 2^{m+1})\,(2^{M+3} - 2^{m+1}) \times [f_4^{(m)} - f_0^{(m)}]/(m+1)!\,\}. \quad \text{(A6)}$$

For higher values of L such a straightforward elimination is not possible, though the formulas for $L = 6$ can be derived by comparing the expressions $I_{1,\,M+6,\,0}$ for $h = 1$, 2, 3 and 6. The equations (A4)–(A6) include the corresponding Newton-Cotes formulas, provided M is formally equated to -1, instead of zero.

The expression

$$I_{111} = I_{100} - \frac{11}{120}h^2(f_r' - f_0') + \frac{1}{120}h(\mu\delta_h f_r - \mu\delta_h f_0) \quad \text{(A7)}$$

follows from identification of h^3f''' with $\mu\delta_h^3 f$ (this is rigorous for quartic polynomials) and elimination of this quantity from I_{130} and I_{103} in (3)–(7).

Formulas of the type I_{L0N} are most easily derived by means of the symbolic operators E and δ

$$E^n f_0 = f_n,\ \tfrac{1}{2}(E - E^{-1}) = \mu\delta,\ E - 2 + E^{-1} = \delta^2. \quad \text{(A8)}$$

Hence a correction term of the form

$$H_2 = f_0 - 2f_1 + f_2 \quad \text{(A9)}$$

can be written as

$$H_2 = (1 - 2E + E^2)f_0 = \frac{(E-1)^2}{E^2 - 1}(f_2 - f_0) = \frac{E - E^{-1}}{E + 2 + E^{-1}}(f_2 - f_0) = \frac{2\mu\delta}{4 + \delta^2}(f_2 - f_0) = \left(\frac{1}{2}\mu\delta - \frac{1}{8}\mu\delta^3 + \frac{1}{32}\mu\delta^5 - + \ldots\right)(f_2 - f_0). \quad \text{(A10)}$$

Similarly

$$H_3 = f_0 - f_1 - f_2 + f_3 = \frac{E - E^{-1}}{E + 1 + E^{-1}}(f_3 - f_0) = \frac{2\mu\delta}{3 + \delta^2}(f_3 - f_0) \quad \text{(A11)}$$

and

$$H_{22} = f_0 - 2f_2 + f_4 = \frac{E - E^{-1}}{E + E^{-1}}(f_4 - f_0) = \frac{2\mu\delta}{2 + \delta^2}(f_4 - f_0). \quad \text{(A12)}$$

Writing the expressions I_{L0N} in the form

$$I_{L0N} = I_{100} + \alpha H_L + \sum_{n=1}^{N}{}' c_n(\mu\delta^n f_L - \mu\delta^n f_0) \quad \text{(A13)}$$

for $L = 2$ or 3, and equating the coefficients of $\mu\delta^n$ obtained with the use of the expansions (A10) or (A11) with those in (5) and (7) up to $N+2$, we obtain the coefficients listed in Table 1. For $L=4$ we have to consider two correction terms $\alpha_1 H_2$ and $\alpha_2 H_{22}$ and equate coefficients up to order $N+4$. This method can, with some loss of simplicity, be extended to $L=5$ and 6.

The general formulas I_{LMN} can similarly be derived symbolically if the differential operators D^m are expanded in terms of $\mu\delta^n$; a list of these expansions is given in Chapter 9 of Kopal's book (1961), where the operator 2θ corresponds to the more widely used D, and in the introduction to Miller's report (1960).

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Book Reviews (*continued from p.* 227)

chapter shows the generality of solutions using dimensionless variables, not only in reducing the work in a particular problem but also for showing the connections between problems from different fields, such as mechanics and electricity, and so making possible the development of analogue computers.

The later chapters in the book are more specific and inevitably reflect the author's interests. A more comprehensive set of illustrations would have needed a team of writers covering other fields, but would have given a more balanced picture of the relations between mathematics and industry. Chapter 4 discusses "spinning in the textile industry," and is followed by one on "Nuclear Reactors," which shows how the partial differential equations are replaced by a large number of simultaneous linear algebraic equations which are adapted to computer solution. Chapter 6 discusses linear programming, an important part of operational research, using the "stepping stone" method and that of "fictitious costs." Chapter 7 shows how much work in aircraft design is solving equations with different boundary conditions corresponding to different profiles, and then discusses how computers have made possible the solutions of the equations of meteorology. The last chapter gives the mathematical analysis of burning wood and relates the analysis to the physics of the process.

Most chapters conclude with one or more references to other books or articles, but a more comprehensive bibliography which could have included other topics from, e.g., the electrical industry, would have been useful. As part of the aim of a series like this is to encourage students to use books, it is unfortunate that no index has been included. Despite this minor criticism the book will be found useful by that increasing body of teachers aware of the importance of relating their mathematics teaching to the problems of modern industry. It should certainly be ordered for all school and college libraries, and many teachers will want to buy a personal copy. P. J. WALLIS.

Symbols, Signals and Noise, by J. R. PIERCE, 1962; 305 pages. (London: *Hutchinson and Co.*, 21s. 0d.)

It is a pleasure to be able to commend a book as highly as this one can be commended. It is an excellent presentation of, and commentary on, the subject of information theory in its various aspects.

After a preamble on the nature of physical and mathematical theories, the book proceeds through the history of electrical communication, the problems of coding, the idea of measuring information and the problem of combating disturbances. It then discusses the application of these ideas in physics, computing, psychology and art, and ends with a brief mention of some recent investigations.

The author is in a senior position in the Bell Telephone Laboratories and has obviously been closely involved in a great deal of this subject. His historical account, which begins with the origin of the Morse code and with the problem of signalling over telegraph cables, is written with obvious human feeling. This is not a mathematical book, and the mathematical reader may find some of the treatment tedious. It is, however, very definitely a pro-mathematical book, in that it stresses throughout the dominating part played by mathematics in the development of the subject, and tries with a good deal of success to give the reader a feel for the things that motivate mathematicians. The author takes great care to indicate just how far he has attempted to present the subject with precision, and at what point he reverts to a description in general terms. He goes out of his way to help the non-mathematical reader to follow as much as possible of the book, even to the extent of giving a miniature textbook of mathematics in an appendix. (This is in fact probably too brief for its purpose, and is interesting rather as a glimpse of mathematics from a new angle for those who can understand it already.) A more useful appendix is a short but excellent glossary.

Many chapters begin with seemingly irrelevant and wasteful digressions, but these always turn out to be written with a worth-while objective. One even finds that the apparently casual examples in the early chapters are designed to prepare the ground for later chapters in the book. The end of each chapter is a clear summary of its main points. In the later chapters, which roam over such fields as psychology and cybernetics, there is much that will interest and intrigue the reader who is concerned with computers; here, however, the connection with information theory often becomes tenuous. It is clear that the simple matter of measuring amounts of information, which is the central concern of information theory, is only a beginning to a study of these other fields.

(*Continued on p.* 245)