Structural Engineering; this was not possible before because of the relatively small size of the Mercury fast store; I hope to report on the results in the near future.

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References

MARTIN, D. W., and TEE, G. J. (1961). "Iterative methods for linear equations with symmetric positive definite matrix", *The Computer Journal*, Vol. 4, p. 242.

LAUGHTON, M. A., and HUMPHREY DAVIES, M. W. "Numerical methods for Power System Load Flow studies". To be published in the *Proceedings* of the *Institution of Electrical Engineers.*

Note on the numerical solution of linear differential equations with constant coefficients

By R. E. Scraton and J. W. Searl

The numerical solution of the differential equation

$$ay'' + by' + cy = f(x) \quad (1)$$

where a, b, c are constants and $f(x)$ is a numerically specified function, can be obtained by a variety of methods. For automatic computation the Runge–Kutta method is normally used, but this may be unstable. The procedure described below provides an alternative method which has been found satisfactory where the Runge–Kutta method has failed.

Suppose that $f(x)$ is tabulated at interval h. In the usual notation, let f_p denote $f(x_0 + ph)$ so that equation (1) may be written

$$a\frac{d^2y}{dp^2} + bh\frac{dy}{dp} + ch^2y = h^2f_p. \quad (2)$$

It is assumed that y_0 and y_0 are known, so that a procedure for determining y_1 and y_1' makes it possible to tabulate both y and y' in a step-by-step manner.

Let the sequence λ_r be defined by the equations

$$\left.\begin{aligned} \lambda_0 &= \frac{1}{a} \\ \lambda_1 &= -\frac{bh}{a^2} \\ a\lambda_{r+2} + bh\lambda_{r+1} + ch^2\lambda_r &= 0, \qquad r \geqslant 0 \end{aligned}\right\} \quad (3)$$

and let

$$U_m(p) = \frac{\lambda_0 p^m}{m!} + \frac{\lambda_1 p^{m+1}}{(m+1)!} + \frac{\lambda_2 p^{m+2}}{(m+2)!} + \dots \quad (4)$$

It is easily verified that

$$a\frac{d^2U_m}{dp^2} + bh\frac{dU_m}{dp} + ch^2U_m = \left\{\begin{array}{ll} 0 & \text{if } m = 0, 1 \\ \dfrac{p^{m-2}}{(m-2)!} & \text{if } m \geqslant 2. \end{array}\right\} \quad (5)$$

If, therefore, f_p can be expanded in the form

$$f_p = A_0 + A_1p + A_2p^2 + A_3p^3 + \dots \quad (6)$$

it can be shown that the solution of (2) satisfying the required initial conditions is

$$\begin{aligned} y = y_0[1 - ch^2U_2(p)] &+ ahy_0'U_1(p) \\ &+ h^2[A_0U_2(p) + A_1.1!U_3(p) \\ &\qquad + A_2.2!U_4(p) + \dots]. \end{aligned} \quad (7)$$

Thus

$$\begin{aligned} y_1 = y_0[1 - ch^2u_2] &+ ahy_0'u_1 \\ &+ h^2(A_0u_2 + A_1.1!\,u_3 + A_2.2!\,u_4\dots) \end{aligned} \quad (8)$$

and

$$\begin{aligned} hy_1' = -\,ch^2u_1y_0 &+ ahy_0'u_0 \\ &+ h^2(A_0u_1 + A_1.1!\,u_2 + A_2 2!\,u_3\dots) \end{aligned} \quad (9)$$

where

$$u_m = U_m(1) = \frac{\lambda_0}{m!} + \frac{\lambda_1}{(m+1)!} + \frac{\lambda_2}{(m+2)!} + \dots$$

The coefficients A_r of equation (6) may be taken from any polynomial interpolation formula. For automatic computation, Lagrangian formulae are appropriate, and a four-point formula will be used as an illustration, viz:

$$\begin{aligned} f_p = &-\frac{1}{6}(p^3 - 3p^2 + 2p)f_{-1} \\ &+ \frac{1}{2}(p^3 - 2p^2 - p + 2)f_0 \\ &- \frac{1}{2}(p^3 - p^2 - 2p)f_1 + \frac{1}{6}(p^3 - p)f_2. \end{aligned} \quad (10)$$

Equations (8) and (9) may then be written

$$y_1 = y_0(1 - ch^2u_2) + ahy_0'u_1 + P_{-1}f_{-1} + P_0f_0 + P_1f_1 + P_2f_2 \quad (11)$$

and

$$y_1' = - ch^2u_1y_0 + ahy_0'u_0 + Q_{-1}f_{-1} + Q_0f_0 + Q_1f_1 + Q_2f_2 \quad (12)$$

where

$$P_{-1} = - h^2\left(\frac{1}{3}u_3 - u_4 + u_5\right)$$

$$Q_{-1} = - h^2\left(\frac{1}{3}u_2 - u_3 + u_4\right)$$

$$P_0 = - h^2\left(u_2 - \frac{1}{2}u_3 - 2u_4 + 3u_5\right)$$

$$Q_0 = h^2\left(u_1 - \frac{1}{2}u_2 - 2u_3 + 2u_4\right)$$

$$P_1 = h^2(u_3 + u_4 - 3u_5)$$

$$Q_1 = h^2(u_2 + u_3 - 3u_4)$$

$$P_2 = - h^2\left(\frac{1}{6}u_3 - u_5\right)$$

$$Q_2 = h^2\left(\frac{1}{6}u_2 - u_4\right).$$

The principal terms in the truncation errors of equations (11) and (12) are $\frac{11}{1,440}\frac{h^6f^{IV}}{a}$ and $\frac{11}{720}\frac{h^6f^{IV}}{a}$, respectively.

It will be appreciated that a certain amount of preliminary computation is necessary to determine the coefficients P_n, Q_n; once these have been evaluated the tabulation of y and y' is very rapid.

Reference

Modern Computing Methods, p. 142. (National Physical Laboratory, Notes on Applied Science No. 16 (H.M.S.O.))

Book Reviews: Numerical Analysis

Approximate Calculation of Integrals, by V. I. Krylov, translated by Arthur H. Stroud, 1962; 357 pages. (New York: *The Macmillan Company*. $7.50. London: *The Macmillan Company, New York*.)

The translator's preface opens with the following paragraph: "This book provides a systematic introduction to the subject of approximate integration, an important part of numerical analysis. Such an introduction was not available previously. The manner in which the book is written makes it ideally suited as a text for a graduate seminar course on this subject." These claims are amply justified; the author deals with the fundamental principles underlying the subject with rigour and with clarity (for which we must also thank the translator). The validity of the additional claim made on the flap, that "this book is a useful reference for practical computations" is open to doubt.

In Part 1, consisting of Chapters 1 to 4, the author discusses Bernoulli numbers and polynomials, orthogonal polynomials, interpolation, and elementary functional analysis. The material is presented in a manner which will make it readily understood by a mathematics graduate, and it is sufficient to equip him to follow the arguments of the remaining parts of the book. Although this preliminary information is all available elsewhere, the student will be grateful for its inclusion here in such a lucid and coherent form. Part 2 deals with the problem of definite integration. The problem considered is that of integrating over a segment (usually finite) the product of a fixed weight function and a "well-behaved" arbitrary function. The result is to be approximated by a linear combination of the values of the arbitrary function at certain "nodes", the coefficients being dependent upon the weight function. This part of the book explores thoroughly the more important aspects of this problem; the various ways in which the nodes might be selected, the special case of periodic integrals, the various forms of representation of the remainder, the possibilities for reducing the remainder by removing or "weakening" singularities, and so on. Particular attention is paid to certain classes of formulae, notably the Newton-Cotes formulae and those of "the highest algebraic degree of precision" (perhaps more commonly known as Gauss-type formulae). The latter are considered at some length with respect to a general weight function, and the most common special cases also receive attention, in particular those in which the weight function takes the form of (i) a constant, (ii) $(b - x)^\alpha (x - a)^\beta$, (iii) e^{-x^2} and (iv) $x^\alpha e^{-x}$. The first two are associated with the finite segment $[a, b]$ and the last with the ranges $(- \infty, \infty)$ and $(0, \infty)$ respectively.

The last four chapters, forming Part 3, are concerned with the computation of indefinite integrals. They cover the use of recurrence relations to evaluate an integral at successive values of the argument (with a full discussion of convergence and stability), finite-difference methods, and methods using intermediate (non-tabular) nodes. Several numerical examples are included.

Appendices A, B and C give numerical values of the nodes and weights for the Gauss integration formulae of most common types, i.e. Gauss-Legendre, Gauss-Hermite and Gauss-Laguerre. These values are all reproduced from other sources.

A few criticisms of the book might be made. It can scarcely be regarded as a satisfactory working manual, because there is no systematic comparison of the various available methods applicable in any given set of circumstances. For instance, in section 7.4 it is observed that an integral of the form $\int_{\infty}^{\infty} e^{-x^2} f(x)\, dx$ can be evaluated by the appropriate Gauss-Hermite formula, and an example shows that in the case $f(x) = J_0(x)$ the use of 10 nodes yields a result which is in error by only about 2.10^{-13}. This result appears less impressive when it is realized that application of the simplest of quadrature formulae, the trapezoidal rule, at an interval of $\frac{1}{2}$ in x, gives a result which is in error by about 2.10^{-15}, using 11 nodes. The high accuracy obtainable for this kind of integral using the trapezoidal rule is well known; reasons for it were discussed by Goodwin (1949). Similar remarks apply to the evaluation of periodic integrals; Fettis (1955) treats the class of integrals for which the trapezoidal rule is highly accurate, by considering Poisson's summation formula.