Eigenvectors of the successive over-relaxation process, and its combination with Chebyshev semi-iteration

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The eigenvector decomposition of the errors of the S.O.R. process is examined, and proofs are given for conjectures which have been published concerning the elementary divisors of the process. The results of numerical experiments with a Chebyshev semi-iterative procedure based on S.O.R. are interpreted in the light of this analysis, and it is concluded that the structure of the eigenvectors of the S.O.R. process makes the process unsuitable for use in Chebyshev semi-iteration. It is demonstrated that a knowledge of the maximum eigenvalue of an interative process is not always adequate for specifying the convergence of such a process—the structure of the eigenvectors can have a profound influence upon the convergence.

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PART I

Figenvalues of Error Operators

1. Introduction

When an iterative process for solving a set of linear algebraic equations is analyzed, attention is usually concentrated upon the eigenvalues of the error operator of the process, since the asymptotic convergence rate of the errors depends on the spectral radius (i.e. the maximum modulus of the eigenvalues) of the error operator. But it can happen that, in some cases, the eigenvalues by themselves give an inadequate specification of the behaviour of the errors, and it may be necessary also to analyze the eigenvectors of the error operator, in order to account for the actual behaviour of the errors.

As an illustration of this, we shall compare two iterative methods for solving a system of linear equations $Ax = b$, where the symmetric matrix A is consistently ordered (Ref. Forsythe and Wasow (1960), p. 244). The Successive Over-Relaxation process (S.O.R.), starting from any initial estimate $x^{(0)}$ for x, produces a sequence of vectors which converge towards x at the same rate (asymptotically) as do the vectors produced by the Chebyshev–Seidel process. But in practice it is found that S.O.R. gives very much better approximations to x than does the Chebyshev–Seidel method, after the same number of iterations from the same initial estimate. We shall show that this is a consequence of the structure of the eigenvectors of the S.O.R. error operator.

In Part I we consider first (in § 3) the eigenvalues and eigenvectors of the error operator of the Simultaneous Displacement Method (S.D.M.). Then in § 4 (which is based on the treatment in § 22.1 of Forsythe and Wasow, 1960) the eigenvalues of the S.O.R. error operator are related to those of the S.D.M. error operator. In § 6 we prove a conjecture made by G. E. Forsythe and W. R. Wasow, that the multiple zero eigenvalue of the error operator for the Seidel process (i.e. S.O.R. with $\omega = 1$) is associated with linear elementary divisors, provided that A is permuted into the so-called "σ_1-ordering." An example is given in § 7 of a matrix (for the finite-difference Dirichlet problem over a rectangle) which does not have σ_1-ordering, and for which the Seidel process has non-linear elementary divisors.

In Part II we construct the eigenvectors of the S.O.R. error operator (§ 8), and show that its elementary divisors are always linear if $\omega \neq 1$. Accordingly, the initial error can be expressed as a linear combination of eigenvectors of the S.O.R. error operator (except possibly when $\omega = 1$), and the coefficients of this linear combination are found in § 9. The eigenvector components of the initial error corresponding to small eigenvalues are examined in detail in § 10. A Chebyshev semi-iterative procedure for accelerating the convergence of the Seidel process is described in § 11, and in § 12 we compare the numerical results for S.O.R. and Chebyshev–Seidel applied to examples of the finite-difference Dirichlet problem over a rectangle. The slow and irregular convergence of the Chebyshev–Seidel process is interpreted as a consequence of the structure of the eigenvectors of the S.O.R. error operator with small eigenvalues.

We conclude (§ 13) that the Chebyshev semi-iterative process based on S.O.R. is best applied with $\omega = 1$ and with σ_1-ordering.

2. Tridiagonal representations of matrices

We shall consider systems of equations

$$Ax = b \tag{2.1}$$

where the $n \times n$ matrix A has "Property A" and is

consistently ordered, i.e. there exists some tridiagonal representation

$$M = \Pi A \Pi^T \tag{2.2}$$

(where Π is a permutation matrix) which is ordered consistently with respect to A (cf. Forsythe and Wasow (1960), p. 243).

Moreover, we shall restrict our attention to systems of equations with matrices M which are themselves diagonally block-tridiagonal. There is no further loss of generality in considering only the tridiagonal representation M rather than A itself, for the eigenvalues and eigenvectors of the error operators of both S.D.M. and S.O.R. are the same for M as for A, apart from permutations within the eigenvectors.

The matrix M is diagonally m-block tridiagonal, i.e. it may be partitioned into the form:

$$M = \begin{bmatrix} D_1 & F_1 & & & & \\ E_1 & D_2 & F_2 & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & E_{m-1} & D_{m-1} & F_{m-1} \\ & & & & E_{m-1} & D_m \end{bmatrix} \tag{2.3}$$

where $m > 1$ and the partitions $D_i (i = 1, \ldots, m)$ are each *diagonal* square submatrices. Separating out the non-zero elements of M which are respectively below, on, and above the diagonal, we get that:

$$M = E + D + F \tag{2.4}$$

where

$$E = \begin{bmatrix} O & & & \\ E_1 & O & & \\ & \ddots & \ddots & \\ & & E_{m-1} & O \end{bmatrix}, \quad D = \begin{bmatrix} D_1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & D_m \end{bmatrix},$$

$$F = \begin{bmatrix} O & F_1 & & \\ & O & \ddots & \\ & & \ddots & F_{m-1} \\ & & & O \end{bmatrix}. \tag{2.5}$$

We assume that all diagonal elements of M are non-zero, so that D^{-1} exists.

3. Error operator of S.D.M.

The error operator K for S.D.M. applied to the matrix M is given by

$$K = -D^{-1}(E + F) \tag{3.1}$$

(cf. Varga (1962), p. 57).

We shall assume from now on that the elementary divisors of K are linear, so that its eigenvectors span n-space. It is easily proved that this condition holds if A is symmetric. The characteristic polynomial of K is

$$\begin{aligned} P(\lambda) &= \det [K - \lambda I] = \det [-D^{-1}(E + F) - \lambda I] \\ &= \det [-D^{-1}(E + \lambda D + F)] \\ &= (-1)^n . \det D^{-1} . \det [E + \lambda D + F]. \end{aligned} \tag{3.2}$$

It can be shown (cf. Forsythe and Wasow (1961), p. 248) that, in view of the structure of M, the characteristic polynomial of K has the form

$$P(\lambda) = \lambda^k Q(\lambda^2) \tag{3.3}$$

where k is some non-negative integer and $Q(x)$ is a polynomial in x of degree $\frac{1}{2}(n - k)$, with $Q(0) \neq 0$. Thus zero is an eigenvalue of K with multiplicity k, and (3.1) shows that if v_0 is any eigenvector of K with zero eigenvalue, then

$$(E + F)v_0 = 0. \tag{3.4}$$

Since we have assumed that all elementary divisors of K are linear, the eigenvectors v_0 must span a space of k dimensions. Therefore the nullity of the matrix $(E + F)$ must be k (cf. Aitken (1956), p. 69), and hence the rank of the matrix $(E + F)$ is $(n - k)$.

Equation (3.2) shows that if $\lambda_i \neq 0$ is an eigenvalue of K with multiplicity μ, then $-\lambda_i$ is also an eigenvalue of K with the same multiplicity. If v_i is an eigenvector of K with eigenvalue λ_i, i.e.

$$-D^{-1}(E + F)v_i = \lambda_i v_i, \tag{3.5}$$

then

$$(E + \lambda_i D + F)v_i = 0. \tag{3.6}$$

4. S.O.R. error operator

We shall now derive an equation connecting an eigenvalue η of the S.O.R. error operator with an eigenvalue λ of the S.D.M. error operator.

The error operator for S.O.R. applied to the matrix M has the following form

$$H = -\left(E + \frac{1}{\omega}D\right)^{-1}\left[\left(1 - \frac{1}{\omega}\right)D + F\right] \tag{4.1}$$

(cf. Forsythe and Wasow (1960), p. 247).

The characteristic polynomial of H is

$$\begin{aligned} T(\eta) &= \det [H - \eta I] \\ &= \det \left[-\left(E + \frac{1}{\omega}D\right)^{-1}(\eta E + \zeta D + F)\right] \\ &= (-1)^n . \left(\det \left[E + \frac{1}{\omega}D\right]\right)^{-1} . \det [\eta E + \zeta D + F] \\ &= (-\omega)^n . (\det D)^{-1} . \det [\eta E + \zeta D + F] \end{aligned} \tag{4.2}$$

where

$$\zeta = \frac{\eta + \omega - 1}{\omega}. \tag{4.3}$$

Define the diagonal matrix

$$S = \begin{bmatrix} I_1 & & & & & \\ & \eta^{1/2}I_2 & & & & \\ & & \eta I_3 & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & \eta^{(m-1)/2}I_m \end{bmatrix} \tag{4.4}$$

where $\eta^{1/2}$ has been chosen from a selected branch of the function $z^{1/2}$; e.g. if η is positive, $\eta^{1/2}$ may be chosen as positive.

Then it follows from the structure of E, D, F and S that:

$$\begin{aligned} T(\eta) &= (-\omega)^n.(\det D)^{-1}.\det S^{-1} \\ &\qquad .\det[\eta E + \zeta D + F].\det S \\ &= (-\omega)^n.(\det D)^{-1}.\det[S^{-1}(\eta E + \zeta D + F)S] \\ &= (-\omega)^n.(\det D)^{-1}.\det[\eta^{1/2}E + \zeta D + \eta^{1/2}F] \\ &= (-\omega)^n.(\det D)^{-1}.\eta^{n/2}.\det[E + \eta^{-1/2}\zeta D + F]. \end{aligned} \tag{4.5}$$

Using (3.2), this gives

$$T(\eta) = \omega^n.\eta^{n/2}.P(\eta^{-1/2}\zeta) \tag{4.6}$$

and (3.3) shows that

$$\begin{aligned} T(\eta) &= \omega^n.\eta^{n/2}.(\eta^{-1/2}\zeta)^k.Q\left(\frac{\zeta^2}{\eta}\right) \\ &= \omega^{n-k}(\eta + \omega - 1)^k\left[\eta^{(n-k)/2}Q\left\{\frac{(\eta+\omega-1)^2}{\omega^2\eta}\right\}\right] \end{aligned} \tag{4.7}$$

where $[\eta^{(n-k)/2}Q\{\zeta^2\eta^{-1}\}]$ is a polynomial in η of degree $(n-k)$ which is non-zero when $\zeta = 0$, unless $\eta = 0$ also. Thus $\eta = 1 - \omega$ is an eigenvalue of H with multiplicity k, and the other $(n-k)$ roots satisfy the equation

$$\eta + \omega - 1 = \eta^{1/2}\omega\lambda \tag{4.8}$$

whose λ is some non-zero eigenvalue of K.

The transformation of the set λ_i to the set η_i (by equation (4.8)) has been exhaustively investigated by Frankel (1950), Young (1954), Kjellberg (1958), Engeli* (1959) and by Forsythe and Wasow (1960). Here we note that each value of λ can be transformed into a *unique* value of η. On the other hand, if (4.8) is regarded as a quadratic equation in $\eta^{1/2}$, its discriminant will vanish when ω satisfies the equation

$$\omega^2\lambda^2 - 4(\omega - 1) = 0 \tag{4.9}$$

in which case both λ and $-\lambda$ will transform into a single value of η. Thus if λ_i satisfies (4.9) and is an eigenvalue of K with multiplicity μ, the corresponding η will have multiplicity 2μ.

* A useful graph of η and ω for various values of λ is given on p. 89.

Equations (4.7) and (4.8) relate the eigenvalues λ_i of K to the eigenvalues η_i of H for the particular tridiagonal representation M of the original matrix, A. Now consider a different tridiagonal representation of A (e.g. $\Pi_3 A \Pi_3^T$), which need not be ordered consistently with respect to A (and hence to M). The λ_i will all be the same as for M, and since the η_i are related to the λ_i by (4.7) and (4.8) the values of the η_i *including their multiplicities* will be the same for $\Pi_3 A \Pi_3^T$ as for M, whether or not these tridiagonal representations of A are ordered consistently with respect to one another. The η_i will, of course, be the same for $\Pi_3 A \Pi_3^T$ as for any other permutation (say, $\Pi_4 A \Pi_4^T$) which is ordered consistently with respect to it.

Consistently ordered matrix	*Tridiagonal representation ordered consistently with respect to the matrix*
A	$M = \Pi A \Pi^T$
$\Pi_4 A \Pi_4^T$	$\Pi_3 A \Pi_3^T$.

Thus we conclude that the eigenvalues of the S.O.R. process applied to a set of equations $Ax = b$ (where A has "Property A") are the same for all permutations of A which are consistently ordered, *whether or not these are ordered consistently with respect to one another.*

On the other hand, we shall show that the corresponding elementary divisors of H can be different for different consistent orderings of A, so that the eigenvectors of H corresponding to an eigenvalue η with multiplicity μ can span a space of μ dimensions for one consistent ordering, but span a space of fewer dimensions for a different consistent ordering.

5. Seidel process

When $\omega = 1$ the S.O.R. process reduces to the Successive Displacement Method, named variously after Seidel, Nekrassov and Liebmann. H can have zero eigenvalues only when $\omega = 1$ (cf. (4.7)), when the characteristic polynomial becomes

$$T(\eta) = \eta^{(n+k)/2}Q(\eta). \tag{5.1}$$

Thus when $\omega = 1$, then $\eta = 0$ is an eigenvalue of H with multiplicity $\frac{1}{2}(n+k)$, i.e. at least half of the eigenvalues are zero. The non-zero eigenvalues are given by (cf. (4.8))

$$\eta_i = \lambda_i^2. \tag{5.2}$$

The corresponding eigenvectors w must satisfy the equation

$$0 = Hw = -(E + D)^{-1}Fw \tag{5.3}$$

$$Fw = 0. \tag{5.4}$$

The maximum number of linearly independent solutions of (5.4) is equal to the nullity of the matrix F (cf. Aitken (1956), p. 71), and hence the number of linearly independent eigenvectors of the Seidel process with zero-eigenvalue is equal to $(n - r)$, where r is the rank of the matrix F.

6. Zero eigenvalues of Seidel process with σ_1-ordering

Any matrix A with "Property A" can be permuted into a diagonally 2-block tridiagonal form, i.e. a permutation matrix Π_2 exists such that

$$M = \Pi_2 A \Pi_2^T = \begin{bmatrix} D_1 & F_1 \\ E_1 & D_2 \end{bmatrix} \tag{6.1}$$

where D_1 and D_2 are diagonal square submatrices.*

Such a permutation of A is called a "σ_1-ordering" (cf. Young (1954), p. 108). In this case the matrices E and F assume the forms

$$E = \begin{bmatrix} 0 & 0 \\ E_1 & 0 \end{bmatrix}, \quad F = \begin{bmatrix} 0 & F_1 \\ 0 & 0 \end{bmatrix}. \tag{6.2}$$

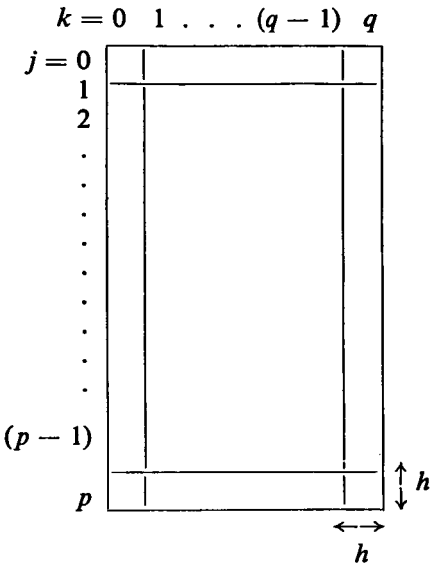
We have seen in § 3 that the rank of $(E + F)$ is equal to $(n - k)$ (cf. (3.6)). It is readily shown that, in view of (6.2), the rank of $(E + F)$ equals the sum of the ranks of E_1 and of F_1.

We shall assume from now on that the matrix A is symmetric. It follows that E_1 must be the transpose of F_1, and hence their ranks are equal. Therefore the rank of F equals the rank of F_1, which is half the rank of $(E + F)$; i.e. the rank of F is $\frac{1}{2}(n - k)$. Therefore the nullity of F is $n - \frac{1}{2}(n - k) = \frac{1}{2}(n + k)$, and hence the number of linearly independent eigenvectors of H with zero eigenvalue is equal to $\frac{1}{2}(n + k)$. But this is equal to the multiplicity of the zero eigenvalue, and hence the elementary divisors associated with zero eigenvalue must be linear.

We shall now construct a system of equations whose matrix is ordered consistently with respect to a diagonally m-block tridiagonal representation where $m > 2$, and shall prove that the multiple zero eigenvalue of the Seidel process for this matrix is associated with non-linear elementary divisors.

7. Dirichlet problem in a rectangle

Consider the Dirichlet problem for the 5-point Laplace operator on a square net drawn over a rectangular region of dimensions $ph \times qh$ (cf. Frankel (1950) and Heller (1959)). Let the rows of the net be numbered $j = 0$ to p, and the columns be numbered $k = 0$ to q.



Let Poisson's equation hold over the rectangle

$$\nabla^2 \phi = S \tag{7.1}$$

where S is known, and let ϕ be known everywhere on the boundary. Denote the values of ϕ and of S at the node (jh, kh) by $\phi_{j,k}$ and $S_{j,k}$ respectively. Then the standard finite-difference approximation to (7.1) holds at every internal node $(0 < j < p,\ 0 < k < q)$, giving the set of $(p - 1)(q - 1)$ equations

$$\phi_{j-1,k} + \phi_{j,k-1} - 4\phi_{j,k} + \phi_{j,k+1} + \phi_{j+1,k} = h^2 S_{j,k} \tag{7.2}$$

$$\left.\begin{aligned} j &= 1, 2, \ldots, p - 1 \\ k &= 1, 2, \ldots, q - 1 \end{aligned}\right\}.$$

If S.D.M. is applied to the $(p - 1)(q - 1)$ equations (7.2), it is readily shown that for all orderings of the equations (in which each equation is solved for the value of ϕ at the corresponding central node) the eigenvectors of the error operator K are given by

$$\epsilon_{j,k}^{(r,s)} = \sin\left(\frac{\pi r j}{p}\right) . \sin\left(\frac{\pi s k}{q}\right) \tag{7.3}$$

$$\left.\begin{aligned} r &= 1, \ldots, p - 1 \\ s &= 1, \ldots, q - 1 \end{aligned}\right\}$$

and the corresponding eigenvalue is (cf. (4) and (7) in Frankel (1950)) given by

$$\lambda^{(r,s)} = \tfrac{1}{2}\left(\cos\frac{\pi r}{p} + \cos\frac{\pi s}{q}\right). \tag{7.4}$$

This shows that $\lambda^{(r,s)} = 0$ when

$$\frac{\pi r}{p} + \frac{\pi s}{q} = \pi. \tag{7.5}$$

Hence, the multiplicity k of the zero eigenvalue of K

* Indeed, the matrix M of (2.3) may be permuted into

$$\Pi_5 M \Pi_5^T = \begin{bmatrix} D_1 & & & F_1 & & \\ & D_3 & & E_2 & F_3 & \\ & & \ddots & & \ddots & \ddots \\ & & & \ddots & & \ddots \\ E_1 & F_2 & & D_2 & & \\ & E_3 & \ddots & & D_4 & \\ & & \ddots & \ddots & & \ddots \end{bmatrix}$$

is equal to the number of pairs of integers (r, s) which satisfy the equation

$$\frac{r}{p} + \frac{s}{q} = 1 \tag{7.6}$$

where $0 < r < p$, $0 < s < q$. Thus $k = 0$ if p and q are co-prime.

Let the internal nodes (and correspondingly the equations) be numbered in the so-called "page-wise" order

$$(j, k) = (1, 1), \ldots, (1, q-1), \ldots, (p-1, 1), \ldots, (p-1, q-1).$$

It is readily shown that the resulting matrix A (of dimensions $(p-1)(q-1) \times (p-1)(q-1)$) has "Property A" and that it is ordered consistently with respect to a tridiagonal representation M in which each successive partition corresponds to the nodes along successive diagonals of the net (cf. Young (1954), p. 108; Forsythe and Wasow (1960), p. 245)*. i.e. the partitions of M correspond to

$$(j, k) = [(1, 1)], [(1, 2), (2, 1)], [(1, 3), (2,2), (3, 1)], \ldots, [(p-1), (q-1)].$$

Thus the page-wise ordering is a consistent ordering, and hence the eigenvalue analysis of § 3 and § 4 is applicable.

With the equations ordered page-wise, the resulting matrix may be written in partitioned form as

$$A = \begin{bmatrix} U & I & & & & & \\ I & U & I & & & & \\ & \ddots & \ddots & \ddots & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & I & U & I \\ & & & & & I & U \end{bmatrix} \tag{7.7}$$

where I is a $(q-1) \times (q-1)$ unit matrix and U is a $(q-1) \times (q-1)$ submatrix

$$U = \begin{bmatrix} -4 & 1 & & & & & \\ 1 & -4 & 1 & & & & \\ & \ddots & \ddots & \ddots & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & 1 & -4 & 1 \\ & & & & & 1 & -4 \end{bmatrix}. \tag{7.8}$$

It is easily seen that without loss of generality we may take $p \geqslant q$.

* Engeli mistakenly asserts (in Engeli *et al.* (1959), p. 87) that the page-wise ordering is not a consistent ordering.

A permutation matrix Π exists such that

$$M = E + D + F = \Pi A \Pi^T \tag{7.9}$$

so that $$A = \Pi^T M \Pi = \Pi^T E \Pi + \Pi^T D \Pi + \Pi^T F \Pi \tag{7.10}$$

where in this case $D = -4I$ and $E = F^T$. Since A and M are ordered consistently with respect to one another, the matrix F of the non-zero elements above the diagonal of M must permute into the matrix of non-zero elements above the diagonal of A. Thus

$$\Pi^T F \Pi = \begin{bmatrix} V & I & & & & \\ & V & I & & & \\ & & \ddots & \ddots & & \\ & & & \ddots & \ddots & \\ & & & & V & I \\ & & & & & V \end{bmatrix} \tag{7.11}$$

where the $(q-1) \times (q-1)$ submatrix V has the form

$$V = \begin{bmatrix} 0 & 1 & & & & \\ & 0 & \ddots & & & \\ & & \ddots & \ddots & & \\ & & & \ddots & \ddots & \\ & & & & 0 & 1 \\ & & & & & 0 \end{bmatrix}. \tag{7.12}$$

The rank of F equals that of $\Pi^T F \Pi$, but this is at least $(p-2)(q-1)$, for $\Pi^T F \Pi$ contains the $(p-2)(q-1) \times (p-2)(q-1)$ minor

$$\begin{bmatrix} I & & & & \\ V & I & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & I & \\ & & & V & I \end{bmatrix} \tag{7.13}$$

which is non-singular, since its determinant $= 1$. Thus the nullity of F is less than q, and hence (cf. § 5) the number of linearly independent eigenvectors of the Seidel process with zero eigenvalue is not more than $q-1$. But the multiplicity of the zero eigenvalue is $\frac{1}{2}(n+k)$ where $k \geqslant 0$ (cf. (5.1)), where $n = (p-1)(q-1)$, so that $\frac{1}{2}(n+k) \geqslant \frac{1}{2}n > (q-1)$ if $p > 3$. Indeed the only cases where the strict inequality $(\frac{1}{2}(n+k) > q-1)$ might not hold would be (*a*) $p = 3$, $q = 3$, (*b*) $p = 3$, $q = 2$ and (*c*) $p = 2$, $q = 2$. It is readily found that the strict inequality holds for case (*a*) ($k = 2$, cf. (7.4)), but that we get equality for case (*b*). Case (*c*) is trivial, since there is only one unknown.

Therefore we have shown that, except in the trivial

cases of 1 or 2 internal nodes, the eigenvectors of the Seidel process applied to (7.2) with zero eigenvalue span a space whose dimensionality is less than the multiplicity of the zero eigenvalue.

Combining this with the results of § 6, we see that we have proved the following conjecture and assertion made by G. E. Forsythe and W. R. Wasow (1960, p. 260), with reference to the Dirichlet problem:

"For one consistent order considered above (first all points of odd parity, then all points of even parity), the eigenvalue 0 seems to be associated only with linear elementary divisors (we have not seen a proof), although its multiplicity is approximately $N/2$. However, if one orders the points of a net by rows (like reading a page of English) the eigenvalue 0 has non-linear divisors of various multiplicities m; for each non-linear divisor of multiplicity m there is a vector X such that $H^m X = 0$ but $H^{m-1}X \neq 0$. This means that for less than m iterations, the eigenvalue 0 does not achieve its asymptotic state of annihilating approximately half the principal directions. Thus an asymptotic definition of the rate of convergence cannot really apply when there are fewer than m iterations, as there may be for large problems."

PART II

8. Eigenvectors of S.O.R. error operator

We shall now construct the eigenvectors of the S.O.R. error operator H with non-zero eigenvalues, on the assumption that A is symmetric.

Let w be an eigenvector of H with eigenvalue η, i.e.

$$Hw = \eta w. \tag{8.1}$$

Then (cf. (4.2))

$$\left[\eta E + \left(\frac{\eta}{\omega} + 1 - \frac{1}{\omega}\right)D + F\right]w = 0. \tag{8.2}$$

Therefore, *provided* that $\eta \neq 0$ (which is always true for $\omega \neq 1$, and is true for $\frac{1}{2}(n-k)$ values of η when $\omega = 1$), S will be non-singular and hence (cf. (4.3)),

$$S^{-1}(\eta E + \zeta D + F)SS^{-1}w = 0 \tag{8.3}$$

from which we get (cf. (4.5))

$$(E + \eta^{-1/2}\zeta D + F)S^{-1}w = 0. \tag{8.4}$$

Hence if η is connected with λ by the relation (4.8) (i.e. $\lambda = \eta^{-1/2}\zeta$), then (cf. (3.6)) $S^{-1}w$ will be an eigenvector of K with eigenvalue λ. Conversely, if v is an eigenvector of K with eigenvalue λ, then

$$w = Sv \tag{8.5}$$

is an eigenvector of H with eigenvalue η.

Although the eigenvalues η of H have been shown to be the same for all consistent orderings of A, equation (8.5) shows that the structure of the eigenvectors w will be different for different tridiagonal representations of A. Indeed the eigenvector v may be partitioned compatibly with M and the corresponding eigenvector of H will be, according to (8.5)

$$w = Sv = \begin{bmatrix} V_{(1)} \\ \eta^{1/2}V_{(2)} \\ \eta V_{(3)} \\ \vdots \\ \eta^{(m-1)/2}V_{(m)} \end{bmatrix}. \tag{8.6}$$

But if $m > 2$ and A is re-ordered into a σ_1-ordering, the eigenvector w now has the form appropriate to $m = 2$

$$w = \begin{bmatrix} V_{(1)} \\ V_{(3)} \\ \vdots \\ \eta^{1/2}V_{(2)} \\ \eta^{1/2}V_{(4)} \\ \vdots \end{bmatrix} \tag{8.7}$$

which is not, in general, simply a permutation of w for the matrix $M = \Pi A \Pi^T$ as in (8.6) (cf. footnote to (6.1)).

We shall now show that, for $\eta \neq 0$, any multiple eigenvalues η correspond to linear elementary divisors. Let η be an eigenvalue of H with multiplicity $\mu > 1$. Then the corresponding $\lambda(= \eta^{-1/2}\zeta)$ is an eigenvalue of K either with multiplicity μ (cf. § 4) or with multiplicity $\frac{1}{2}\mu$. This latter case will occur when (cf. (4.9))

$$\lambda = \pm \frac{2}{\omega}\sqrt{(\omega - 1)} \tag{8.8}$$

in which event $-\lambda$ will also be an eigenvalue of K with multiplicity $\frac{1}{2}\mu$.

In either case we may construct a set of μ linearly independent eigenvectors $v_1, \ldots, v_\mu$ of K, since all elementary divisors of K are linear (cf. § 3). The vectors $Sv_1, \ldots, Sv_\mu$ are all eigenvectors of H with eigenvalue η, and, if they were linearly dependent so that

$$c_1 Sv_1 + \ldots + c_\mu Sv_\mu = 0, \tag{8.9}$$

then premultiplication by S^{-1} would give

$$c_1 v_1 + \ldots + c_\mu v_\mu = 0 \tag{8.10}$$

which contradicts the linear independence of the v_i, unless $c_1 = \ldots = c_\mu = 0$. Therefore the μ eigen-

vectors of $\boldsymbol{H}$ corresponding to the μ-fold eigenvalue η are linearly independent, and the corresponding elementary divisors must be linear.

Combining this result with that of § 6, we obtain the following theorem:

Theorem: *The error operator of the S.O.R. process applied to a symmetric consistently ordered matrix A has linear elementary divisors if $\omega \neq 1$. Furthermore, when $\omega = 1$ the elementary divisors are linear provided that A has σ_1-ordering. For other consistent orderings with $\omega = 1$, non-linear elementary divisors may be associated with the multiple zero-eigenvalue.*

Thus the eigenvectors w of $\boldsymbol{H}$ will always form a complete basis for vectors in n-space, except possibly when $\omega = 1$ and $m > 2$.

9. Error expansion in eigenvectors of the S.O.R. error operator

When a stationary iterative process (e.g. S.D.M. or S.O.R.) is applied for solving a system of equations $Ax = b$, the convergence of the sequence of current estimates towards the true solution x is most conveniently investigated by analyzing the initial error vector into eigenvectors of the error operator of the iterative process. This can always be done, unless the error operator has non-linear elementary divisors. The theorem of the previous paragraph shows that this eigenvector analysis can be performed with S.O.R., provided that $\omega \neq 1$.

We shall consider first the case of linear elementary divisors of $\boldsymbol{H}$, and shall examine the other case subsequently.

For an initial estimate $x^{(0)}$ of the solution x of (2.1), the initial error vector is

$$e^{(0)} = x - x^{(0)}. \tag{9.1}$$

Expand this in terms of eigenvectors w_i of $\boldsymbol{H}$.

$$e^{(0)} = \sum_{i=1}^{n} b_i w_i. \tag{9.2}$$

If the vector produced after r cycles of the S.O.R. process is $x^{(r)}$, then

$$e^{(r)} = x - x^r = \sum_{i=1}^{n} b_i \eta_i^r w_i. \tag{9.3}$$

In order to evaluate the coefficients b_i we shall construct the eigenvectors of $\boldsymbol{H}^T$, which are biorthogonal to those of $\boldsymbol{H}$ (cf. Faddeeva (1959), p. 41). If z_j is an eigenvector of $\boldsymbol{H}^T$ with eigenvalue η_j, equation (4.1) shows that

$$\eta_j z_j = -\left\{\left(1 - \frac{1}{\omega}\right)\boldsymbol{D} + \boldsymbol{E}\right\}\left\{\boldsymbol{F} + \frac{1}{\omega}\boldsymbol{D}\right\}^{-1} z_j \tag{9.4}$$

where we have used the fact that $\boldsymbol{E} = \boldsymbol{F}^T$. Define the vector

$$t_j = \left\{\boldsymbol{F} + \frac{1}{\omega}\boldsymbol{D}\right\}^{-1} z_j \tag{9.5}$$

so that

$$z_j = \left\{\boldsymbol{F} + \frac{1}{\omega}\boldsymbol{D}\right\} t_j. \tag{9.6}$$

Then (9.4) and (9.5) give

$$-\left\{\left(1 - \frac{1}{\omega}\right)\boldsymbol{D} + \boldsymbol{E}\right\} t_j = \eta_j \left\{\boldsymbol{F} + \frac{1}{\omega}\boldsymbol{D}\right\} t_j. \tag{9.7}$$

Let $\eta_j \neq 0$. Then

$$\left\{\frac{1}{\eta_j}\boldsymbol{E} + \left[\frac{1}{\eta_j}\left(1 - \frac{1}{\omega}\right) + \frac{1}{\omega}\right]\boldsymbol{D} + \boldsymbol{F}\right\} t_j = 0 \tag{9.8}$$

or

$$\left\{\xi \boldsymbol{E} + \left(\frac{\xi}{\rho} + 1 - \frac{1}{\rho}\right)\boldsymbol{D} + \boldsymbol{F}\right\} t_j = 0 \tag{9.9}$$

where

$$\xi = \frac{1}{\eta_j}, \quad \frac{1}{\rho} = 1 - \frac{1}{\omega}. \tag{9.10}$$

Comparison of (9.9) with (8.2) shows that t_j must be an eigenvector of the error operator of S.O.R. applied to the matrix $\boldsymbol{M}$, with ω replaced by ρ and with an eigenvalue $\xi = \frac{1}{\eta}$. We conclude that, by analogy with (8.5),

$$t_j = \boldsymbol{S}^{-1} v_j \tag{9.11}$$

where $\boldsymbol{S}^{-1}$ has been used, since η has been replaced by $\xi = \eta^{-1}$. Hence (cf. (9.6))

$$z_j = \left\{\boldsymbol{F} + \frac{1}{\omega}\boldsymbol{D}\right\} \boldsymbol{S}^{-1} v_j. \tag{9.12}$$

The biorthogonality relation between the eigenvectors of $\boldsymbol{H}$ and of $\boldsymbol{H}^T$ means that $z_j^T w_i = 0$ unless $i = j$, in which event

$$\begin{aligned} z_i^T w_i &= v_i^T \boldsymbol{S}^{-1}\left\{\boldsymbol{E} + \frac{1}{\omega}\boldsymbol{D}\right\} w_i \\ &= v_i^T \boldsymbol{S}^{-1}\left\{\boldsymbol{E} + \frac{1}{\omega}\boldsymbol{D}\right\} \boldsymbol{S} v_i \\ &= v_i^T \left\{\eta_i^{-1/2}\boldsymbol{E} + \frac{1}{\omega}\boldsymbol{D}\right\} v_i \end{aligned} \tag{9.13}$$

in view of the structure of $\boldsymbol{E}$, $\boldsymbol{D}$ and $\boldsymbol{S}$.

Normalize the eigenvectors v_i of $\boldsymbol{K}$ so that

$$v_i^T \boldsymbol{D} v_i = -1. \tag{9.14}$$

Then (9.13) and (9.14) show that

$$z_i^T w_i = \eta_i^{-1/2} v_i^T \boldsymbol{E} v_i - \frac{1}{\omega}. \tag{9.15}$$

But (3.6) shows that

$$\begin{aligned} 0 &= v_i^T(\boldsymbol{E} + \lambda_i \boldsymbol{D} + \boldsymbol{F}) v_i \\ &= v_i^T \boldsymbol{E} v_i + \lambda_i v_i^T \boldsymbol{D} v_i + v_i^T \boldsymbol{F} v_i \\ &= v_i^T \boldsymbol{E} v_i - \lambda_i + (v_i^T \boldsymbol{E} v_i)^T \\ &= -\lambda_i + 2 v_i^T \boldsymbol{E} v_i \end{aligned} \tag{9.16}$$

since $v_i^T E v_i$ is a scalar. Equations (9.15) and (9.16) show that

$$z_i^T w_i = -\frac{1}{\omega} - \frac{\lambda_i}{2\eta_i^{1/2}} \tag{9.17}$$

and equation (4.8) enables us to eliminate λ_i, giving

$$z_i^T w_i = -\frac{1}{\omega} + \frac{\eta_i + \omega - 1}{2\eta_i \omega} = -\frac{(1 + \eta_i - \omega)}{2\eta_i \omega}. \tag{9.18}$$

Premultiplication of (9.2) by z_i^T produces

$$z_i^T e^{(0)} = \sum_{j=1}^{n} b_j z_i^T w_j = b_i z_i^T w_i. \tag{9.19}$$

Thus we have an explicit expression for the coefficient b_i

$$\begin{aligned} b_i &= \frac{z_i^T e^{(0)}}{z_i^T w_i} = \frac{-2\eta_i \omega}{1 - \omega + \eta_i} . (z_i^T e^{(0)}) \\ &= \frac{-2\eta_i \omega}{1 - \omega + \eta_i} . \left\{ v_i^T S^{-1} \left(E + \frac{1}{\omega} D \right) e^{(0)} \right\} \\ &= \frac{-2\eta_i}{1 - \omega + \eta_i} . \{ v_i^T S^{-1} (\omega E + D) e^{(0)} \}. \end{aligned} \tag{9.20}$$

Since η_i is a known function of ω for any fixed λ_i (cf. (4.8)), equation (9.20) gives b_i for any ω.

In particular, if $\omega = 1$ and $\eta_i \neq 0$ we get

$$b_i = -2 v_i^T S^{-1} (E + D) e^{(0)}. \tag{9.21}$$

When $\omega = 1$, only $\frac{1}{2}(n - k)$ eigenvalues are non-zero. Thus if A has σ_1-ordering we may rewrite (9.2) as

$$e^{(0)} = \sum_{i=1}^{\frac{1}{2}(n-k)} b_i w_i + \sum_{\frac{1}{2}(n-k)+1}^{n} b_i w_i \tag{9.22}$$

where all the eigenvectors w_i with non-zero eigenvalue have been grouped into the first term on the right of (9.22), and their coefficients b_i may be evaluated by (9.21). But the remaining term is a linear combination of eigenvectors of H with zero eigenvalue, which must itself be an eigenvector with eigenvalue zero. Thus, if the vector u is defined by the equation

$$e^{(0)} = \sum_{i=1}^{\frac{1}{2}(n-k)} b_i w_i + u \tag{9.23}$$

where all w_i are eigenvectors of H with non-zero eigenvalues, then (9.24) expresses $e^{(0)}$ as a sum of $(\frac{1}{2}(n - k) + 1)$ eigenvectors of H. All terms in this eigenvector expansion may be evaluated, since w_i is given by (8.5), b_i by (9.21), and then u is given by (9.23) itself. A single cycle of the Seidel process will annihilate u.

On the other hand if $\omega = 1$ and A does not have σ_1-ordering, non-linear elementary divisors could be associated with the $\frac{1}{2}(n + k)$-fold zero eigenvalue of H (cf. § 7). In that event the eigenvectors w_i do not form a complete basis for n-space so that (9.2) is invalid. Rather, if ν dimensions have been "lost" owing to non-linear divisors, an expansion of a general n-vector $e^{(0)}$ will be of the form (cf. Faddeeva (1959), p. 53)

$$e^{(0)} = \sum_{i=1}^{\nu} c_i x_i + \sum_{i=\nu+1}^{n} b_i w_i \tag{9.24}$$

where the x_i are "principal vectors" of various grades up to m_1, and the w_i are linearly independent eigenvectors. The derivation of the expression (9.21) for the coefficients b_i of those w_i for which $\eta_i \neq 0$ is still valid, but it is no longer true that the remainder after these have been subtracted from $e^{(0)}$ will itself be an eigenvector (with zero-eigenvalue). The complications arising from non-linear elementary divisors associated with $\eta = 0$ are such as to make a complete analysis (based on (9.24)) of the Seidel process an exceedingly difficult task in such circumstances, especially when the Seidel process is combined with a Chebyshev semi-iterative procedure as in § 11 and § 12. We note, however, that all the x_i components of $e^{(0)}$ (together with the w_i corresponding to $\eta_i = 0$) will be annihilated by m_1 iterations of the Seidel procedure, where m_1 is the maximum order of any non-linear elementary divisor (cf. § 7).

In order to gain a clearer picture of what happens when $\omega = 1$, we shall consider $\omega \neq 1$ so that (9.2) and (9.20) are strictly valid, and shall let $\omega \to 1$, paying particular attention to very small values of η.

10. Error expansion into eigenvectors with small eigenvalues

As $\omega \to 1$, the η_i corresponding to any particular λ_i^2 will approach zero either if $\lambda_i = 0$, or if $\eta_i^{1/2}$ is the smaller of the pair of values associated with λ_i^2 by the equation (4.8). In either event, (8.6) shows that w_i approaches the form

$$\lim_{\eta_i \to 0} w_i = (\lim_{\eta_i \to 0} S) V_i = \begin{bmatrix} v_{i(1)} \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}. \tag{10.1}$$

Similarly, we get that

$$\lim_{\eta \to 0} \eta_i^{(m-1)/2} v_i^T S^{-1} = [0 \ldots 0 \quad v_{i(m)}^T]. \tag{10.2}$$

Consider first the case $\lambda_i = 0$, so that $\eta_i = 1 - \omega$ and hence the coefficient in (9.20) is

$$\frac{-2\eta_i}{1 - \omega + \eta_i} = -1 \tag{10.3}$$

for any $\omega \neq 1$. Then (9.20), (10.2) and (10.3) show that

$$\begin{aligned} \lim_{\substack{\omega \to 1 \\ \lambda_i = 0}} (1 - \omega)^{(m-1)/2} b_i &= -[0 \ldots 0 \quad v_{i(m)}^T][E + D] e^{(0)} \\ &= -\{ v_{i(m)}^T E_{m-1} e_{(m-1)} + v_{i(m)}^T D_m e_{(m)} \} \end{aligned} \tag{10.4}$$

where $e_{(m-1)}$ and $e_{(m)}$ denote the partitions of $e^{(0)}$ corresponding to D_{m-1} and D_m. Thus if $\omega \simeq 1$, (10.4) shows that

$$b_i = 0\left((1 - \omega)^{(1-m)/2}\right). \tag{10.5}$$

Since w_i remains finite as $\omega \to 1$ (cf. (10.1)) the contribution $b_i w_i$ is itself a vector of the order

$$0\left((1 - \omega)^{(1-m)/2}\right).$$

More precisely the mth partition of $b_i w_i$ remains finite, the $(m-1)$th partition is of order $O((1-\omega)^{-1/2})$, . . ., and the first partition is of order

$$O\big((1-\omega)^{(1-m)/2}\big).$$

Secondly, we consider the case $\lambda_i \neq 0$ and $\omega \neq 1$, with η_i being the smaller of the two values of η corresponding to λ_i^2. According to (4.8),

$$\frac{1-\omega}{\eta_i^{1/2}} = \eta_i^{1/2} - \omega\lambda_i \tag{10.6}$$

$$\therefore \quad \frac{2\eta_i}{1-\omega+\eta_i} = \frac{2\eta_i^{1/2}}{\dfrac{1-\omega}{\eta_i^{1/2}} + \eta_i^{1/2}}$$

$$= \frac{2\eta_i^{1/2}}{2\eta_i^{1/2} - \omega\lambda_i} \approx \frac{-2\eta_i^{1/2}}{\lambda_i} \tag{10.7}$$

as $\eta_i \to 0$, $\omega \to 1$. But (10.6) shows that

$$\frac{\eta_i^{1/2}}{1-\omega} \approx -\frac{1}{\lambda_i} \tag{10.8}$$

as $\eta_i \to 0$, $\omega \to 1$. Hence $\eta_i = O((1-\omega)^2)$, and

$$\lim_{\substack{\eta_i \to 0 \\ \omega \to 1}} (1-\omega)^{-1} \frac{2\eta_i}{1-\omega+\eta_i} = \frac{2}{\lambda_i^2}. \tag{10.9}$$

Equations (9.20), (10.2) and (10.9) show that

$$\lim_{\substack{\omega \to 1 \\ \eta_i \to 0}} b_i(1-\omega)^{-1} . \eta_i^{(m-1)/2}$$

$$= \frac{-2}{\lambda_i^2}[0 \ldots 0 \quad v_{i(m)}^T][\boldsymbol{E}+\boldsymbol{D}]e^{(0)}$$

$$= \frac{-2}{\lambda_i^2}\{v_{i(m)}^T E_{m-1} e_{(m-1)} + v_{i(m)}^T D_m e_{(m)}\}. \tag{10.10}$$

Using (10.8), the term on the left becomes

$$\lim_{\substack{\omega \to 1 \\ \eta_i \to 0}} b_i(1-\omega)^{-1} . \frac{(1-\omega)^{m-1}}{(-\lambda_i)^{m-1}}$$

$$= (-\lambda_i)^{1-m} \lim_{\omega \to 1} (1-\omega)^{m-2} b_i \tag{10.11}$$

$$\therefore \lim_{\omega \to 1} b_i(1-\omega)^{m-2}$$

$$= (-1)^m \lambda_i^{m-3} 2\{V_{i(m)}^T E_{m-1} e_{(m-1)} + V_{i(m)}^T D_m e_{(m)}\}. \tag{10.12}$$

Hence, as $\omega \to 1$ and $\eta_i \to 0$, (10.12) shows that

$$b_i = O\big((1-\omega)^{2-m}\big) \tag{10.13}$$

or in view of (10.8)

$$b_i = O\big(\eta_i^{1-m/2}\big) \tag{10.14}$$

and since w_i remains finite as $\omega \to 1$ (cf. (10·1)), the contribution $b_i w_i$ will itself be of this order. Thus, the contribution $b_i w_i$ for the $\frac{1}{2}(n-k)$ values of η_i which tend to zero as $\omega \to 1$ (with $\lambda_i \neq 0$) will become indefinitely large as $\omega \to 1$, if $m > 2$.

Also, (10.5) shows that the k contributions $b_i w_i$ corresponding to $\lambda_i = 0$ will always become indefinitely large as $\omega \to 1$, since always $m > 1$.

Thirdly, we note that when $\eta_i \neq 0$ and $\omega = 1$, b_i is given by (9.21) and accordingly if λ_i is small (so that $\eta_i = \lambda_i^2$ is very small), then

$$b_i \simeq -2\,\eta_i^{(1-m)/2}[0 \ldots 0 \quad v_{i(m)}^T][\boldsymbol{E}+\boldsymbol{D}]e^{(0)}$$

$$= -2\lambda_i^{1-m}\{v_{i(m)}^T E_{m-1} e_{(m-1)} + v_{i(m)}^T D_m e_{(m)}\} \tag{10.15}$$

whilst (8.6) shows that

$$w_i = \boldsymbol{S}v_i \simeq \begin{bmatrix} v_{i(1)} \\ 0 \\ . \\ . \\ . \\ 0 \end{bmatrix}. \tag{10.16}$$

Thus in each of the three circumstances under which η_i can be very small, the coefficient b_i is almost independent of the first $(m-2)$ partitions of $e^{(0)}$ (cf. (10.4), (10.12) and (10.15)), and the eigenvector w_i approaches the form of (10.16). Moreover, the coefficient b_i will (in general) be smaller when $\boldsymbol{M}$ has σ_1-ordering than when it is permuted into any tridiagonal representation with $m > 2$ (cf. (10.13), (10.14) and (10.15)).

11. Chebyshev semi-iteration applied to the S.O.R. process

Provided that the eigenvalues λ_i of the error operator of an iterative process are known to lie on a specified segment of the real axis $(-1 < -l \leqslant \lambda_i \leqslant l < 1)$, and provided that all elementary divisors are linear, then it is known that suitably normalized Chebyshev polynomials in the error operator will reduce the errors much more rapidly than does the basic iterative process itself. (cf. Golub and Varga (1961); Rutishauser, p. 31 in Engeli *et al.* (1959)). Such a procedure may be called a "Chebyshev semi-iterative process," or "Chebyshev acceleration of a basic iterative procedure." On the other hand, if any eigenvalues of the error operator are complex, in general no semi-iterative process can be guaranteed to improve the convergence (cf. Varga, (1957)).

If a symmetric matrix $\boldsymbol{A}$ is consistently ordered and if all eigenvalues λ_i of $\boldsymbol{K}$ are known to lie in the range

$$-1 < -\rho \leqslant \lambda_i \leqslant \rho < 1 \tag{11.1}$$

then it can be shown from (4.8) that all eigenvalues η_i of $\boldsymbol{H}$ are real and non-negative for all ω in the range $0 < \omega \leqslant \omega^*$, where ω^* is some number in the range $1 < \omega^* < 2$. For $\omega > \omega^*$, some or all of the η_i are complex. The maximum η_i decreases as ω increases from 0 to ω^* (and beyond), and accordingly, the rate of convergence of a Chebyshev semi-iterative procedure based on S.O.R. will increase as ω increases. Since $\omega = \omega^* > 1$ gives the smallest maximum η_i for which all η_i are real, this would be the optimum value to choose. But ω^* depends on the smallest value of λ_i^2 (cf. Engeli *et al.* (1959), p. 89) which is generally unknown. How-

ever, it usually happens that $\omega^* - 1 \ll 1$, so that little could be gained by increasing ω beyond 1. Therefore ω would generally be chosen as 1.

Provided that $\omega \neq 1$ (or if $\omega = 1$, provided that M has σ_1-ordering) the eigenvector expansion (9.2) is valid and the Chebyshev semi-iteration can be justified. If a polynomial $p_\nu(H)$ is generated by the νth stage of a semi-iterative process, then the error at the νth stage is given by

$$\epsilon^{(\nu)} = \sum_{i=1}^{n} b_i p_\nu(\eta_i) w_i \tag{11.2}$$

(cf. (2.5) in Golub and Varga, 1961).

If we apply our knowledge that all η_i satisfy the relation

$$0 \leqslant \eta_i \leqslant l < 1 \tag{11.3}$$

then the optimal polynomials to use are given by

$$p_\nu^*(x) = \frac{T_\nu\left(\frac{2x}{l} - 1\right)}{T_\nu\left(\frac{2}{l} - 1\right)} \tag{11.4}$$

where the Chebyshev polynomials are defined by

$$\left.\begin{aligned} T_\nu(z) &= \cos(\nu \cos^{-1} z) \text{ if } |z| \leqslant 1 \\ T_\nu(z) &= \operatorname{ch}(\nu \operatorname{ch}^{-1} z) \text{ if } |z| > 1 \end{aligned}\right\} \tag{11.5}$$

from which it follows that

$$T_\nu(z) = \tfrac{1}{2}\{[z + \sqrt{(z^2 - 1)}]^\nu + [z + \sqrt{(z^2 - 1)}]^{-\nu}\}. \tag{11.6}$$

The definition (11.5) shows that $-1 \leqslant T_\nu(z) \leqslant 1$ if $-1 \leqslant z \leqslant 1$ for all ν, and accordingly

$$\frac{-1}{T_\nu\left(\frac{2}{l} - 1\right)} \leqslant p_\nu^*(\lambda_i) \leqslant \frac{1}{T_\nu\left(\frac{2}{l} - 1\right)} \tag{11.7}$$

for all λ_i in the range $0 \leqslant \lambda_i \leqslant l$. Thus by the νth stage of the Chebyshev process using (11.4), every projection $b_i w_i$ has been multiplied by a factor lying anywhere between $-S_\nu$ and S_ν, where S_ν is defined as

$$\begin{aligned} S_\nu = \frac{1}{T_\nu\left(\frac{2}{l} - 1\right)} &= \frac{2}{\left\{\frac{2}{l} - 1 + \sqrt{\left[\left(\frac{2}{l} - 1\right)^2 - 1\right]}\right\}^\nu + \left\{\frac{2}{l} - 1 + \sqrt{\left[\left(\frac{2}{l} - 1\right)^2 - 1\right]}\right\}^{-\nu}} \\ &< 2\left\{\frac{2}{l} - 1 + \sqrt{\left[\left(\frac{2}{l} - 1\right)^2\right] - 1}\right\}^{-\nu} \\ &= 2\left\{\frac{1 - \sqrt{(1 - l)}}{1 + \sqrt{(1 - l)}}\right\}^\nu. \end{aligned} \tag{11.8}$$

Consider the effect of Chebyshev acceleration of S.O.R. when ω is taken close to 1. The value of $l\,(= \max \eta_i)$ varies continuously as ω varies through 1, and hence S_ν will vary continuously for all ω in the range $0 < \omega \leqslant \omega^*$. When $\omega = 1$, then $l = \lambda_1^2$, where $\lambda_1 = \max \lambda_i$ (cf. (5.2)); so that $S_\nu \to 1/T_\nu\left(\frac{2}{\lambda_1^2} - 1\right)$ as $\omega \to 1$. On the other hand, we have seen (cf. (10.5) and (10.13)) that as $\omega \to 1$, those $\frac{1}{2}(n + k)$ contributions $b_i w_i$ for which $\eta_i \to 0$ become indefinitely large if $m > 2$ (and even for $m = 2$ if $\lambda_i = 0$), particularly in the first few partitions of w_i. Therefore $e^{(\nu)}$, which is the sum of all contributions, could contain elements which are very much larger than S_ν times elements of $e^{(0)}$. Thus, for ω close to 1 we expect that a Chebyshev semi-iterative procedure based on S.O.R. will retain large errors in the first few partitions, and these errors will decay slowly and erratically.

It is interesting to compare the convergence of the Chebyshev–Seidel method with that of S.O.R. Equation (11.8) shows that every projection of the error lies within bounds which decay during each cycle by a factor f, where

$$f = \frac{1 - \sqrt{(1 - \lambda_1^2)}}{1 + \sqrt{(1 - \lambda_1^2)}} \tag{11.9}$$

when the Chebyshev–Seidel process is applied to the matrix M. But it can be shown (cf. Forsythe and Wasow (1960), p. 256) that when S.O.R. is applied to the same matrix M the optimal value of ω is exactly $\omega_0 = 1 + f$, in which event *all* eigenvalues η of H have modulus equal to f. Thus the decay factor η_i for each contribution in the case of optimized S.O.R. has modulus equal to the decay factor of the bounds within which each contribution is contained in the case of the Chebyshev–Seidel process.* From this we might expect the two processes to converge at about the same rate, but the example in § 12 will show how far this is from being true. The behaviour of the Chebyshev–Seidel process will be interpreted in the light of the eigenvector analysis of § 9 and § 10.

12. Numerical experiments

The Deuce program GE01T for solving the Dirichlet problem by S.O.R. was modified† so as to perform Chebyshev acceleration of the Seidel process.

The version of Chebyshev acceleration used in these programs was slightly different from that discussed in § 11, inasmuch as it used only the knowledge that

$$-1 < l \leqslant \eta_i \leqslant l < 1 \tag{12.1}$$

without taking advantage of the more stringent inequality (11.3). In this case the appropriate sequence of poly-

* The contributions will not be the same in these two cases. Indeed with Chebyshev-Seidel the eigenvectors w_i are those of S.O.R. with $\omega = 1$, whereas in the other they are those of S.O.R. with $\omega = \omega_0$ (cf. (8.6)).

† Mr. B. A. Carré kindly assisted in this modification of his program.

nomials is given (cf. Golub and Varga (1961), p. 149) by

$$\tilde{p}_\nu(x) = \frac{T_\nu\left(\frac{x}{l}\right)}{T_\nu\left(\frac{1}{l}\right)} \qquad (12.2)$$

so that every projection $b_i w_i$ is multiplied by a factor lying between

$$\frac{-1}{T_\nu\left(\frac{1}{l}\right)} \text{ and } \frac{+1}{T_\nu\left(\frac{1}{l}\right)}$$

after ν stages of the Chebyshev semi-iterative method.*

It can be shown, from a comparison of $T_\nu\left(\frac{1}{l}\right)$ and $T_\nu\left(\frac{2}{l} - 1\right)$, that when $1 - l \ll 1$ the bounds within which the contributions are contained by this process (12.2) decay by a factor approximately equal to $f^{1/\sqrt{2}}$, so that approximately $\sqrt{2}$ times as many iterations are needed to reduce these bounds by any specified factor as are needed with (11.4). Apart from this reduction in the convergence rate, the previous argument applies without change.

The model problem which was investigated consisted of solving the standard 5-point approximation to Laplace's equation over a $p \times q$ rectangle, with ϕ fixed at zero everywhere on the boundary (cf. § 7). The solution is, of course, $\phi = 0$ everywhere, so that at any stage the current estimate is equal to the error ($\times -1$). As an initial estimate, ϕ was taken as 1 at all internal nodes.

The eigenvalues $\lambda^{(r,s)}$ of $\boldsymbol{K}$ are given by (7.4), and the non-zero eigenvalues of $\boldsymbol{H}$ are the squares of these: $\eta^{(r,s)} = (\lambda^{(r,s)})^2$. Therefore the maximum eigenvalue l of the Seidel process is equal to

$$l = (\lambda^{(1,1)})^2 \simeq 1 - \frac{\pi^2}{2}\left(\frac{1}{p^2} + \frac{1}{q^2}\right). \qquad (12.3)$$

The Seidel process was performed with pagewise ordering of the nodes (cf. § 7), so that the matrix $\boldsymbol{A}$ was ordered consistently with respect to a tridiagonal representation $\boldsymbol{M}$ in which each successive diagonal partition corresponds to nodes along a diagonal of the net. Thus, after every cycle of the Seidel process (and hence of the Chebyshev semi-iterative procedure) the current vector will be exactly the same as though S.O.R. had been applied to the tridiagonal representation $\boldsymbol{M}$, for which $m = p + q - 3$.

The projections $b_i w_i$ of $e^{(0)}$ corresponding to zero eigenvalue are annihilated after m_1 iterations of the Seidel processes, where m_1 is the maximum order of any non-linear elementary divisor (cf. § 7). All other projections will be multiplied by a factor not greater than l by every iteration, so that by the νth iteration every projection will have been multiplied by a factor lying between 0 and l^ν (provided that $\nu \geqslant m_1$). Table 1 gives N_L in the fourth column, where N_L is that value of ν which makes $l^\nu = 0{\cdot}5 \times 10^{-4}$ for the values of p and q as given in the first two columns. Numerical experiments showed that the number of iterations required to reduce the error itself to less than $0{\cdot}5 \times 10^{-4}$ everywhere on the net was only slightly greater than this number N_L which multiplies every projection by a factor $\leqslant 0{\cdot}5 \times 10^{-4}$.

The fifth column gives N_c, which is defined as the number ν of iterations of the Chebyshev process required to make $1/T_\nu(1/l) \leqslant 0{\cdot}5 \times 10^{-4}$, so that every eigenvector component (but not necessarily the principal vector components) will have been multiplied by a factor lying between $-0{\cdot}5 \times 10^{-4}$ and $+0{\cdot}5 \times 10^{-4}$. But the number of iterations which empirically were found necessary to reduce the error itself below $0{\cdot}5 \times 10^{-4}$ everywhere is given in the final column, and it is seen that this is considerably greater than N_c.

Table 1

p	q	l	N_L	N_c	ACTUAL NUMBER OF ITERATIONS REQUIRED OF CHEBYSHEV PROC.
5	4	0·574	18	8	13
5	5	0·651	24	11	16
7	6	0·780	41	15	26
9	8	0·869	71	20	40
9	9	0·884	86	21	49
12	12	0·933	149	29	$\geqslant$ 40
12	13	0·938	165	31	$\geqslant$ 40

For the larger meshes convergence was extremely slow, with elements of the error still as large as $\pm 0{\cdot}3$ by the 36th iteration. In every case, the errors tended to concentrate near the corner $(j, k) = (0, 0)$.

As an example, Table 2 shows the errors of the 28th, 32nd and 36th iterates for the case $(p, q) = (12, 12)$, where the initial error was everywhere 1 and the errors are given with 4 decimal places.

Considering the eigenvalues of $\boldsymbol{H}$, we observe that the smallest non-zero η_l for the problem is given by $(r, s) = (1, 10)$, for which (cf. (7.4))

$$\eta_l = (\lambda^{(1,10)})^2 \simeq 0{\cdot}0025. \qquad (12.4)$$

The coefficient b_l corresponding to this is given by (9.21), but since η_l is quite small we may use the approximate expression (10.15). This approximate expression involves only the 20th and 21st partitions of $e^{(0)}$, since $m = 12 + 12 - 3 = 21$.

The 20th partition contains the nodes $(j, k) = (10, 11)$ and $(11, 10)$, whilst the 21st partition contains the single node $(j, k) = (11, 11)$. Each of these 3 elements of $e^{(0)}$

* The Chebyshev acceleration was performed by means of a 3-term recurrence relation equivalent to (2.9) in Golub and Varga (1961).

Table 2

$\nu = 28$

− 940	− 596	1180	1404	− 705	−1632	− 439	476	395	120	11
− 596	1071	1071	664	1879	82	958	379	78	87	−22
1180	1757	844	1950	728	1548	121	627	−192	15	18
1404	− 664	−1950	1055	1891	718	1031	83	230	54	1
705	−1879	728	1891	− 979	−1142	604	418	− 70	− 57	− 9
−1632	− 82	1548	− 718	−1142	854	427	−342	− 95	23	7
− 439	958	− 121	−1031	604	427	− 482	− 45	111	6	− 3
476	379	− 627	83	418	− 342	− 45	159	− 27	− 16	1
395	− 78	− 192	230	− 70	− 95	111	− 27	− 23	8	0
120	− 87	15	54	− 57	23	6	− 16	8	0	− 1
11	− 22	8	1	− 9	7	− 3	− 3	0	− 1	0

$\nu = 32$

−1011	−2872	− 542	2467	1091	−1135	− 914	− 37	177	83	16
−2872	−1503	2191	1127	−1522	− 809	438	338	30	− 29	−11
− 542	2191	650	−2228	− 467	1236	335	−268	−132	− 14	3
2467	1127	−2228	− 399	1664	69	− 651	− 74	89	27	2
1091	−1522	− 467	1664	− 113	− 895	157	238	− 3	− 19	− 3
−1135	− 809	1236	69	− 895	287	303	−121	− 47	5	2
− 914	438	335	− 651	157	303	− 92	− 45	39	3	− 1
− 37	338	− 268	− 74	238	− 121	− 45	59	− 6	− 5	0
177	30	− 132	89	− 3	− 47	39	− 6	− 8	2	0
83	− 29	− 14	27	− 19	5	3	− 5	2	0	0
16	− 11	3	2	− 3	2	− 1	0	0	0	0

$\nu = 36$

2158	−2182	−2591	1457	1911	− 174	− 673	−197	33	34	9
−2182	−3263	985	1964	− 533	− 792	79	185	39	− 6	− 4
−2591	985	1660	−1343	− 922	626	345	− 74	− 63	− 12	0
1457	1964	−1343	−1030	946	− 301	− 308	− 75	28	11	1
1911	− 533	− 922	946	226	− 503	− 2	108	7	− 6	− 1
− 174	− 792	626	301	− 503	50	159	− 36	− 20	1	0
− 673	79	345	− 308	− 2	159	− 66	− 24	13	1	0
− 197	185	− 74	− 75	108	− 36	− 24	21	− 1	− 2	0
33	39	− 63	28	7	− 20	13	− 1	− 3	1	0
34	− 6	− 12	11	− 6	1	1	− 2	1	0	0
9	− 4	0	1	− 1	1	0	0	0	0	0

(together with the other 118) is equal to 1. The eigenvector v_l is given by (7.3) with $(r, s) = (1, 10)$, and hence the un-normalized $V^T_{l(m)}$ for this case is the scalar

$$v^T_{l(21)} = \sin\left(\frac{11\pi}{12}\right).\sin\left(\frac{10 \times 11\pi}{12}\right). \qquad (12.5)$$

Next the vector must be normalized so that $v_l^T D v_l = -1$, (cf. (9.14)), where in this case $D = -4I$ (cf. § 7). From (7.3), we get that

$$\begin{aligned} -v_l^T D v_l &= 4 \sum_{j=1}^{p-1} \sum_{k=1}^{q-1} (\epsilon^{(r,s)}_{j,k})^2 \\ &= 4 \sum_{j=1}^{p-1} \sin^2 \frac{\pi r j}{p} \sum_{k=1}^{q-1} \sin^2 \frac{\pi s k}{q} \\ &= 4 . \frac{p}{2} . \frac{q}{2} = pq \end{aligned} \qquad (12.6)$$

(cf. Whittaker and Robinson (1948), p. 263).

Dividing the eigenvectors by $\sqrt{(pq)}$ in order to normalize them, we get that the 21st partition of v_l corresponding to $\eta_l = (\lambda^{(1,10)})^2$ is the scalar

$$\frac{\sin\left(\frac{11\pi}{12}\right).\sin\left(\frac{10 \times 11\pi}{12}\right)}{\sqrt{(12 \times 12)}} \simeq -0{\cdot}011. \qquad (12.7)$$

The partitions E_{20} and D_{21} of M are [1 1] and [−4], respectively. Thus (10.5) gives us

$$\begin{aligned} b_l &\simeq -2\eta_l^{(1-21)/2} \times -0{\cdot}011 \times (1 + 1 - 4) \\ &= -0{\cdot}044 \times (0{\cdot}0025)^{-10} \simeq -5 \times 10^{24}. \end{aligned} \qquad (12.8)$$

The structure of w_l may conveniently be represented by a rectangular array of its elements written at the corresponding nodes of the net. Here, for brevity, the (j, k) element of $\epsilon^{(r,s)}$ (cf. (7.3)) divided by $\sqrt{(pq)}$ (to normalize it) has been represented by $[j, k]$.

$$w_l = \begin{bmatrix} [1,1] & \eta_l^{1/2}[1,2] & \eta_l[1,3] & \cdots & \eta_l^5[1,11] \\ \eta_l^{1/2}[2,1] & \eta_l[2,2] & \eta_l^{3/2}[2,3] & & \vdots \\ \eta_l[3,1] & \eta_l^{3/2}[3,2] & \eta_l^2[3,3] & & \vdots \\ \vdots & & & & \vdots \\ \eta_l^5[11,1] & \cdots & & & \eta_l^{10}[11,11] \end{bmatrix} \qquad (12.9)$$

Evaluating this for $\eta_l = (\lambda^{(1,10)})^2$, using (12.4) and (12.8), we find that the component of the initial error corresponding to *unit* error at each of the three nodes (10, 11), (11, 10) and (11, 11) is almost independent of the initial error at the other nodes, and has the form

$$b_l w_l \simeq \begin{bmatrix} -5\times10^{22} & 4\times10^{21} & -6\times10^{20} & & \\ -4\times10^{21} & 4\times10^{20} & & & \\ -3\times10^{20} & & & & \\ & & & & \\ & & & & \\ & & & & 5\times10^{-3} \end{bmatrix} \qquad (12.10)$$

In the Seidel process each projection $b_l w_l$ is multiplied by η_l in the course of every successive iteration; and since small values of η_l are associated with such remarkably "unbalanced" projections as in (12.10), those projections $b_l w_l$ containing very large elements will be rapidly reduced in magnitude during the first few iterations of the Seidel process.

In contrast to this, by the νth stage of a Chebyshev process (12.2) every projection $b_l w_l$ has been multiplied by a factor $\tilde{p}_\nu(\eta_l)$ which can lie anywhere between $-1/T_\nu(1/l)$ and $+1/T_\nu(1/l)$. In fact, the projections with very small η_l tend to be amplified in comparison with the others when ν is even ($\nu = 2\mu$, say), for then

$$T_\nu(0) = \cos\left(2\mu\frac{\pi}{2}\right) = (-1)^{\mu-1} \qquad (12.11)$$

so that $\tilde{p}_{2\mu}(0) = \pm 1/T_{2\mu}(1/l)$. Hence for any integer N, if η_l is sufficiently small, $\tilde{p}_\nu(\eta_l)$ will be almost as large as possible for $\nu = 2, 4, 6, \ldots, 2N$. In our particular example the initial errors were everywhere equal to 1, and Table 1 shows that every projection will have been multiplied by a factor $< 0{\cdot}5 \times 10^{-4}$ by the 29th iteration. However, we note that the contribution $b_l w_l$

to $e^{(0)}$ has its first element approximately equal to -5×10^{22} (cf. (12.10)), and when $\nu = 32$ we get that $T_\nu\left(\frac{\eta_i}{l}\right) = -0{\cdot}997$. Thus the first element of the corresponding projection of $e^{(32)}$ will be:

$$\frac{T_\nu\left(\frac{\eta_i}{l}\right)}{T_\nu\left(\frac{1}{l}\right)} \times -5 \times 10^{22} \simeq \frac{-0{\cdot}997}{2 \times 10^4} \times -5 \times 10^{22} \simeq 2 \times 10^{18}.$$

In this manner, those projections $b_i w_i$ which correspond to small η_i will be reduced less efficiently than the others generally are,* until ν becomes so large that the equation $T_\nu(x) = 0$ has positive roots smaller than all non-zero η_i. Thus in contrast to the Seidel process where the "unbalanced" projections $b_i w_i$ with small η_i are selectively reduced, the Chebyshev semi-iterative process tends to make them predominate.

The error is the sum of reduced projections $\tilde{p}_\nu(\eta_i) b_i w_i$, and since the reduced projections contain such very large elements (e.g. 2×10^{18} in our example), it is not surprising that their sum contains elements much greater than $0{\cdot}5 \times 10^{-4}$, which is the figure that might be expected from a naïve analysis of the rate of convergence.

Thus the pattern of the errors displayed in Table 2 is explicable, even without invoking the complications arising from non-linear elementary divisors of $\boldsymbol{H}$. When $\boldsymbol{H}$ does have non-linear elementary divisors, the effect of Chebyshev semi-iteration upon the principal vector projections of the error is very difficult to analyze. But we have already noted in § 11 (between (11.8) and (11.9)) that when $0 < |1 - \omega| \ll 1$, the $\frac{1}{2}(n + k)$ projections with small η (for which $\eta \to 0$ as $\omega \to 1$) will be similar in structure to those discussed above (i.e. those with small η when $\omega = 1$). Hence when $\omega \neq 1$ (but ω is close to 1) we expect that by the νth stage the reduced projections corresponding to very small η will add to give elements of $e^{(\nu)}$ very much larger than $1/T_\nu(1/l)$ times elements of $e^{(0)}$, particularly in the first few partitions. If ν is bounded† the same conclusion will hold for $\omega = 1$, by continuity.

13. Conclusions

The convergence properties of an iterative procedure for solving a set of linear equations are not adequately specified by an asymptotic decay factor for bounds containing the error (or for bounds containing the terms of a decomposition of the error). Two different iterative procedures with equal (or nearly equal) asymptotic decay factors (such as Chebyshev–Seidel and S.O.R., applied to a consistently ordered matrix) can give errors of quite different orders of magnitude after the same number of stages of each procedure, starting from the same initial estimate.

When a Chebyshev semi-iterative procedure is based upon the S.O.R. process for a consistently ordered positive-definite matrix, the Seidel process ($\omega = 1$) should be used to minimize the asymptotic decay factor (and also to simplify the computations). Also the matrix should be ordered consistently with respect to σ_1-ordering, for the structure of the eigenvectors of the error operator of the S.O.R. process is such that the eigenvector projections of the initial error will generally contain elements very much larger than the elements of the error itself, but this discrepancy will be less for σ_1-ordering ($m = 2$) than for any permutation into another tridiagonal representation with $m > 2$. Moreover, with σ_1-ordering the errors of the S.O.R. procedure (and of the Chebyshev semi-iteration based on it) can always be analyzed in terms of eigenvectors of the error operator, whereas otherwise this is not possible for $\omega = 1$ so that it is then much more difficult to interpret the errors. A further consequence is that if a single cycle of the Seidel process is applied (before Chebyshev semi-iteration) at least half of the eigenvector components of the error will be annihilated in the case of σ_1-ordering. Moreover, with any consistent ordering this single cycle will greatly reduce in magnitude those troublesome eigenvector contributions with small eigenvalues.

It should be noted, however, that the "Cyclic Chebyshev" method described in § 4 of Golub and Varga (1961) gives the same asymptotic decay factor as Chebyshev–Seidel, but it requires only a single storage area instead of the two needed by Chebyshev–Seidel. Their method is a Chebyshev semi-iterative process based on S.D.M. applied to a matrix with σ_1-ordering, in which only odd-numbered iterates of $x_{(1)}$ and even-numbered iterates of $x_{(2)}$ are computed. Since it is based on S.D.M. their method will be free of the difficulties arising from the structure of the eigenvectors of the S.O.R. error operator. Thus our investigation confirms their verdict (Golub and Varga (1961), p. 148) that for positive-definite matrices with "Property A," the cyclic Chebyshev semi-iterative method is the best systematic iterative technique known.

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* But $T_\nu(0) = 0$ when ν is odd, so that such projections will be almost eliminated for every odd iteration.

† We do not let $\nu \to \infty$ here, since the asymptotic behaviour as $\nu \to \infty$ with fixed ω depends on the Jordan canonical form, which varies continuously with $\omega \neq 1$ but changes discontinuously at $\omega = 1$ itself. Thus the convergence of the S.O.R. results to those of the Seidel process as $\omega \to 1$ is not uniform for all ν.

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Book Reviews

Annual Review in Automatic Programming, Volume 3. Edited by R. GOODMAN, 1963; 360 pages. (Oxford: *Pergamon Press*, 80s.)

The third volume in this series contains, like its predecessors, a collection of independent papers of which all but two can be placed into one or other of the two groups, scientific or commercial. The aims of the previous two volumes have been adhered to, and if it is thought that there are omissions from the present volume, one should remember that the interesting ideas are not always easy for the Editor to chaperone into print. Furthermore, the present volume does reflect the direction in which a great deal of thought was being led at the time of collecting these papers.

On the scientific side all the emphasis is on ALGOL, and the subject really seems to have received justice in this volume. The commercial papers are well introduced by being preceded by an excellent summary of four commercial languages, each of which lays claim to being on a par with COBOL.

The omission of a Preface, as was present in the previous volume, is thought to be a sad loss, but it may be a wise move since the hopes expressed in it on the occasion of its previous appearance cannot be said to have been satisfactorily fulfilled. One feels too, that as more people enter this field, the guiding hand of the Preface could set the scene against which the true achievements can be seen in all their glory.

As to the individual papers, the one describing "A Multi-Pass Translation Scheme for ALGOL 60" was in vogue at the time of writing and is still of commanding interest since it pertains in detail to one of the country's fastest computers. The scheme is reported to be adaptable to any machine but it transpires that a machine having logic similar to that described would be almost essential. The author's thumb-nail description of his own computer on one sheet is quite an achievement, but had it been twice the size the reader would have been pleased. For anyone who has the task of writing an ALGOL translator with optimization, these 44 pages will prove very absorbing, and it is clear that a great deal of thought has been given to the snares and pitfalls which can occur when compiling a language of the scope and complexity of ALGOL. The parallel scheme for a fast "load and go" compiler deserves a mention in the paper as a reviewer feels that many readers would like to pursue the two schemes side by side, as, for instance, they have been presented somewhat differently at a recent symposium.

The paper on "The Compiler Compiler" is presented as being a sequel to previous publications. This need not deter would-be explorers although the previous digestion of past presentations of the same subject will make the journey much easier. If one wishes to criticize the authors, and there are four of them, for the apparently heavy going, let him be reminded that the scheme really does work. This paper deals with the manner of definition of a language in phrase-structure terms, thereby the allowing of the construction of a compiler for that language. Although one of the longer papers in the volume, it is well laid out and makes for clear reading. Some remarks on the more mundane aspects, such as size of program and difficulties of application of the technique, would not have been out of place at the end of the paper.

By far the longest paper in the book deals with an American idea, "Jovial—A Programming Language for Real-time Command Systems". The author prevents the reader's interest from waning by giving trivial yet helpful examples of each concept and, whereas the main theme may be above the skyscrapers, these examples are real-life facts. This paper, whilst having a welcomed introduction, lacks an exposition of the manner in which the author is going to conduct us through the undergrowth of commands in the language. This is but a small defect, however, in a paper which blows a wind of fresh air through the sheets of ALGOL.

The commercial languages are ably and more or less fully reviewed by A. d'Agapeyeff and associates. He compares COBOL and FACT, both of which had a good airing in Vol. 2, and the I.B.M. Commercial Translator and the English counterpart NEBULA with each other. The present authors take each of these main languages in turn and discuss their relative merits, unlike the similar theme which was handled in Vol. 2 subject-wise. One must mention that RAPIDWRITE, CLEO and FILECODE are then described in this informative review. This paper has the advantage of being short and concise, and well deserves its place. In passing, one can note that it again bemoans the

[*Continued on page* 270