Elementary divisors of the Liebmann process

By G. A. Miles, K. L. Stewart and G. J. Tee

When the Liebmann process is applied (with pagewise ordering) to the finite-difference Dirichlet problem for Poisson's equation within a rectangle, the multiple zero eigenvalue of the process is associated with non-linear elementary divisors. The eigenvectors with zero eigenvalue are constructed explicitly, and bounds are deduced for the number of iterations which are needed to annihilate all components of the initial error which are associated with zero eigenvalues.

1. Introduction

It has been shown (Tee, 1963) that when the Dirichlet problem for the finite-difference Poisson equation over a square net is solved by the Liebmann (or Seidel) process with "chessboard" ordering, the multiple zero eigenvalue of the error operator is associated with linear elementary divisors, i.e. the eigenvectors of the error operator span the full number of dimensions. Hence the current error at each stage may be analyzed into eigenvectors of the Liebmann error operator, and all those components of the initial error associated with zero eigenvalue will be annihilated by one cycle of the process. On the other hand, if the Liebmann process is applied with "pagewise" ordering on a square net over a rectangle, it was shown that the multiple zero eigenvalue has non-linear elementary divisors, so that the eigenvectors do not form a complete basis. Those components of the initial error associated with an elementary divisor of order m (with zero eigenvalue) will not be annihilated before m iterations have been performed, so that it would be of interest to know the maximum order m_1 of any elementary divisor, since after m_1 iterations the error of the current estimate can be expressed as a linear combination of eigenvectors of the Liebmann error operator (with non-zero eigenvalues).

In this paper we shall construct explicitly the eigenvectors with zero eigenvalues, and obtain bounds for the number of iterations needed to annihilate all components of the error associated with the multiple zero eigenvalue.

2. Explicit construction of eigenvectors with zero eigenvalue

If the Liebmann process is applied with pagewise ordering to the equations for the 5-node Laplace operator over a square net with $(p-1) \times (q-1)$ internal nodes (and Dirichlet boundary conditions), any eigenvector $\boldsymbol{u}$ of the error operator $\boldsymbol{H}$ must satisfy the equation

$$\boldsymbol{G}\boldsymbol{u} = \boldsymbol{0} \tag{2.1}$$

where (Tee, 1963) $\boldsymbol{G}$ is a compound matrix with $(p-1)$ compound rows and columns:

$$\boldsymbol{G} = \begin{bmatrix} \boldsymbol{V} & \boldsymbol{I} & & & \\ & \boldsymbol{V} & \boldsymbol{I} & & \\ & & \ddots & \ddots & \\ & & & \boldsymbol{V} & \boldsymbol{I} \\ & & & & \boldsymbol{V} \end{bmatrix}, \tag{2.2}$$

$\boldsymbol{I}$ is the $(q-1)$th-order unit matrix, and the $(q-1) \times (q-1)$ submatrix $\boldsymbol{V}$ has the structure

$$\boldsymbol{V} = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{bmatrix}. \tag{2.3}$$

We may assume, without loss of generality, that $p \leqslant q$.*

Partition $\boldsymbol{u}$ conformably with $\boldsymbol{G}$ into $(p-1)$ subvectors $\boldsymbol{u}_j$ with $(q-1)$ elements in each. Then (2.1) shows that

$$\left.\begin{aligned} \boldsymbol{V}\boldsymbol{u}_1 + \boldsymbol{u}_2 &= \boldsymbol{0} \\ \boldsymbol{V}\boldsymbol{u}_2 + \boldsymbol{u}_3 &= \boldsymbol{0} \\ &\cdots\cdots \\ \boldsymbol{V}\boldsymbol{u}_{p-2} + \boldsymbol{u}_{p-1} &= \boldsymbol{0} \\ \boldsymbol{V}\boldsymbol{u}_{p-1} &= \boldsymbol{0} \end{aligned}\right\}. \tag{2.4}$$

In view of the structure of $\boldsymbol{V}$ (cf. (2.3)), it follows from the last equation in (2.4) that $\boldsymbol{u}_{p-1}$ has the structure

$$\boldsymbol{u}_{p-1} = \begin{bmatrix} \alpha_{p-1} \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{2.5}$$

where α_{p-1} is arbitrary.

* This is opposite to the assumption used in Tee (1963).

The $(p-2)$th equation of (2.4) must therefore have the general solution

$$u_{p-2} = \begin{bmatrix} \alpha_{p-2} \\ -\alpha_{p-1} \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}, \tag{2.6}$$

where α_{p-2} is arbitrary. Continuing in this manner, we see that the general solution for the ith partition of $\boldsymbol{u}$ is:

$$u_i = \begin{bmatrix} \alpha_i \\ -\alpha_{i+1} \\ \alpha_{i+2} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ (-1)^{p-1-i}\,\alpha_{p-1} \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} \quad 1 \leqslant i \leqslant p-1. \tag{2.7}$$

Thus the general solution $\boldsymbol{u}$ contains $p-1$ arbitrary constants, and indeed it may be written in the form

$$\boldsymbol{u} = \alpha_1 \boldsymbol{r}_1 + \ldots + \alpha_{p-1}\boldsymbol{r}_{p-1} \tag{2.8}$$

where the vectors $\boldsymbol{r}_i$ may each be partitioned into $(p-1)$ vectors of $(q-1)$ elements, as follows:

$$\boldsymbol{r}_1 = \begin{bmatrix} \boldsymbol{e}_1 \\ \boldsymbol{0} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \boldsymbol{0} \end{bmatrix}, \quad \boldsymbol{r}_2 = \begin{bmatrix} -\boldsymbol{e}_2 \\ \boldsymbol{e}_1 \\ \boldsymbol{0} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \boldsymbol{0} \end{bmatrix}, \quad \boldsymbol{r}_3 = \begin{bmatrix} \boldsymbol{e}_3 \\ -\boldsymbol{e}_2 \\ \boldsymbol{e}_1 \\ \boldsymbol{0} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \boldsymbol{0} \end{bmatrix}, \ldots, \boldsymbol{r}_{p-1} = \begin{bmatrix} (-1)^{p-2}\boldsymbol{e}_{p-1} \\ (-1)^{p-3}\boldsymbol{e}_{p-2} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ -\boldsymbol{e}_2 \\ \boldsymbol{e}_1 \end{bmatrix}, \tag{2.9}$$

where $\boldsymbol{e}_i$ is the ith column of the $(q-1)$th-order unit matrix. Moreover, the vectors $\boldsymbol{r}_i$ are clearly mutually orthogonal. Indeed, it follows from (2.9) that

$$\left.\begin{aligned} \boldsymbol{r}_i^T\boldsymbol{r}_j &= 0 \quad \text{if } i \neq j \\ \boldsymbol{r}_j^T\boldsymbol{r}_j &= j. \end{aligned}\right\} \tag{2.10}$$

Lower bound for m_1

Equations (2.8) and (2.10) show that the eigenvectors $\boldsymbol{u}$ of $\boldsymbol{G}$ with zero eigenvalue span a space of exactly $(p-1)$ dimensions, for which the vectors $\frac{1}{\sqrt{j}}\boldsymbol{r}_j\,(j = 1, \ldots, p-1)$ form an orthonormal basis. Hence the zero eigenvalue of $\boldsymbol{H}$ is associated with eigenvectors spanning exactly $(p-1)$ dimensions $(p \leqslant q)$, although the multiplicity of the zero eigenvalue is $\frac{1}{2}[(p-1)(q-1)+k]$, where $k \geqslant 0$ is the number of pairs of integers (r, s) which satisfy the equation

$$\frac{r}{p} + \frac{s}{q} = 1 \tag{2.11}$$

with $0 < r < p$ and $0 < s < q$ (cf. equation (5.1) and (7.6) of Tee (1963)).

This is a refinement of the result in § 7 of Tee (1963), where it was shown only that the number of linearly independent eigenvectors with zero eigenvalue is less than p and less than q.

The number of elementary divisors of $\boldsymbol{H}$ equals the dimensionality of the space spanned by eigenvectors with zero eigenvalue (cf. Faddeeva (1959), p. 52), which in this case is $p-1$. But the sum of the orders of the elementary divisors equals the multiplicity of the zero eigenvalue, which is $\frac{1}{2}[(p-1)(q-1)+k]$. If we assume that $m_i < \frac{1}{2}(q-1)$ for every elementary divisor m_i, then the sum of the orders would be

$$\sum_{i=1}^{p-1} m_i < \tfrac{1}{2}(p-1)(q-1) \leqslant \tfrac{1}{2}[(p-1)(q-1)+k]. \tag{2.12}$$

This contradiction shows that the maximum order m_1 must satisfy the inequality

$$m_1 \geqslant \tfrac{1}{2}(q-1) \tag{2.13}$$

which implies that

$$m_1 \geqslant \tfrac{1}{2}(p-1). \tag{2.14}$$

3. Minimum polynomial

If λ is an eigenvalue of a matrix, then the multiplicity of λ as a root of the minimum polynomial is equal to the largest order of any elementary divisor corresponding to λ (cf. Halmos (1958), p. 114).

In view of this theorem, we shall investigate the multiplicity of the zero root of the minimum polynomial of $\boldsymbol{H}$. Let n be the order of $\boldsymbol{H}$. The minimum polynomial of $\boldsymbol{H}$ is

$$M(\eta) = \frac{|\boldsymbol{H} - \eta\boldsymbol{I}|}{D(\eta)} \tag{3.1}$$

where $D(\eta)$ is the h.c.f. of all $(n-1)$th-order minors of $|\boldsymbol{H} - \eta\boldsymbol{I}|$ (cf. Faddeeva (1959), p. 52). If the original matrix $\boldsymbol{M}$ is represented in the form

$$\boldsymbol{M} = \boldsymbol{E} + \boldsymbol{D} + \boldsymbol{F} \tag{3.2}$$

where $\boldsymbol{E}$, $\boldsymbol{D}$ and $\boldsymbol{F}$ contain the non-zero elements of $\boldsymbol{A}$ which are respectively below, on, and above the diagonal, then the characteristic polynomial of $\boldsymbol{H}$ is

$$\begin{aligned} T(\eta) = |\boldsymbol{H} - \eta\boldsymbol{I}| &= (-1)^n|\boldsymbol{D}|^{-1}|\eta\boldsymbol{E} + \eta\boldsymbol{D} + \boldsymbol{F}| \\ &= \eta^{\frac{1}{2}(n+k)}\,Q(\eta) \end{aligned} \tag{3.3}$$

where $Q(\eta)$ is a polynomial of order $\frac{1}{2}(n-k)$, with $Q(0) \neq 0$ (cf. Tee (1963), equations (4.2) and (5.1)). Furthermore, every diagonal element of $\boldsymbol{D}$ is -4, so that

$$(-1)^n|\boldsymbol{D}|^{-1} = 4^{-n}. \tag{3.4}$$

All elementary divisors of $\boldsymbol{H}$ are linear except for those corresponding to $\eta = 0$. Hence the minimal polynomial can differ from $T(\eta)$ only by a power of η, so that it follows from (3.1) that $D(\eta)$ must simply be a power of η.

The elements of the matrix $\operatorname{adj}(\boldsymbol{H} - \eta\boldsymbol{I})$ are all the cofactors (i.e. minors with appropriate signs) of $(\boldsymbol{H} - \eta\boldsymbol{I})$ with order $(n-1)$. Therefore, $D(\eta)$ will be the h.c.f. of all the elements of $\operatorname{adj}(\boldsymbol{H} - \eta\boldsymbol{I})$. For any matrix $\boldsymbol{A}$,

$$\boldsymbol{A}(\operatorname{adj}\boldsymbol{A}) = (\operatorname{adj}\boldsymbol{A})\boldsymbol{A} = |\boldsymbol{A}|\boldsymbol{I}. \tag{3.5}$$

Hence, using (3.5), (3.3) and (3.4) we get:

$$\begin{aligned} \operatorname{adj}(\boldsymbol{H} - \eta\boldsymbol{I}) &= |\boldsymbol{H} - \eta\boldsymbol{I}|(\boldsymbol{H} - \eta\boldsymbol{I})^{-1} \\ &= T(\eta)[-(\boldsymbol{E}+\boldsymbol{D})^{-1}\boldsymbol{F} - \eta\boldsymbol{I}]^{-1} \\ &= T(\eta)[-(\boldsymbol{E}+\boldsymbol{D})^{-1}(\eta\boldsymbol{E} + \eta\boldsymbol{D} + \boldsymbol{F})]^{-1} \\ &= -\,T(\eta)(\eta\boldsymbol{E} + \eta\boldsymbol{D} + \boldsymbol{F})^{-1}(\boldsymbol{E}+\boldsymbol{D}) \\ &= -\,4^{-n}|\eta\boldsymbol{E} + \eta\boldsymbol{D} + \boldsymbol{F}| \\ &\qquad (\eta\boldsymbol{E} + \eta\boldsymbol{D} + \boldsymbol{F})^{-1}(\boldsymbol{E}+\boldsymbol{D}) \\ &= -\,4^{-n}(\operatorname{adj}(\eta\boldsymbol{E} + \eta\boldsymbol{D} + \boldsymbol{F}))(\boldsymbol{E}+\boldsymbol{D}). \end{aligned} \tag{3.6}$$

Let η^x be a common factor of every element of the matrix $\operatorname{adj}(\eta\boldsymbol{E} + \eta\boldsymbol{D} + \boldsymbol{F})$. Then (3.6) shows that η^x is also a factor of the elements of the matrix $\operatorname{adj}(\boldsymbol{H} - \eta\boldsymbol{I})$. Conversely, since $(\boldsymbol{E}+\boldsymbol{D})$ is non-singular we get:

$$\operatorname{adj}(\eta\boldsymbol{E} + \eta\boldsymbol{D} + \boldsymbol{F}) = -\,4^n(\operatorname{adj}(\boldsymbol{H} - \eta\boldsymbol{I}))(\boldsymbol{E}+\boldsymbol{D})^{-1} \tag{3.7}$$

so that if η^y is a common factor of all the elements of $\operatorname{adj}(\boldsymbol{H} - \eta\boldsymbol{I})$, then η^y is also a common factor of the elements of the matrix $\operatorname{adj}(\eta\boldsymbol{E} + \eta\boldsymbol{D} + \boldsymbol{F})$. Therefore $D(\eta)$, which is the h.c.f. of all the elements of $\operatorname{adj}(\boldsymbol{H} - \eta\boldsymbol{I})$, is the highest power of η which is a common factor of the elements of the $n \times n$ matrix $\operatorname{adj}(\eta\boldsymbol{E} + \eta\boldsymbol{D} + \boldsymbol{F})$.

Since $\boldsymbol{E} + \boldsymbol{D} + \boldsymbol{F}$ is a tridiagonal representation (ordered consistently with respect to pagewise ordering), it follows that

$$(\eta\boldsymbol{E} + \eta\boldsymbol{D} + \boldsymbol{F}) = \boldsymbol{S}(\eta^{1/2}\boldsymbol{E} + \eta\boldsymbol{D} + \eta^{1/2}\boldsymbol{F})\boldsymbol{S}^{-1} \tag{3.8}$$

where

$$\boldsymbol{S} = \begin{bmatrix} \boldsymbol{I}_1 & & & & & \\ & \eta^{1/2}\boldsymbol{I}_2 & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & \eta^{\frac{1}{2}(m-1)}\boldsymbol{I}_m \end{bmatrix}. \tag{3.9}$$

and $m = p + q - 3$ (cf. equation (4.4) and (4.5) in Tee (1963)). Therefore,

$$\begin{aligned} &\operatorname{adj}(\eta\boldsymbol{E} + \eta\boldsymbol{D} + \boldsymbol{F}) \\ &= (\operatorname{adj}\boldsymbol{S}^{-1})(\operatorname{adj}(\eta^{1/2}\boldsymbol{E} + \eta\boldsymbol{D} + \eta^{1/2}\boldsymbol{F}))(\operatorname{adj}\boldsymbol{S}) \\ &= \eta^{\frac{1}{2}(n-1)}(\operatorname{adj}\boldsymbol{S}^{-1})(\operatorname{adj}(\boldsymbol{E} + \eta^{1/2}\boldsymbol{D} + \boldsymbol{F}))(\operatorname{adj}\boldsymbol{S})\ldots \end{aligned} \tag{3.10}$$

since every element of $\operatorname{adj}(\eta^{1/2}\boldsymbol{E} + \eta\boldsymbol{D} + \eta^{1/2}\boldsymbol{F})$ is a determinant of order $(n-1)$, and a factor of $\eta^{1/2}$ can be extracted from each element of every determinant. But (cf. (3.5))

$$\operatorname{adj}\boldsymbol{S} = |\boldsymbol{S}|\boldsymbol{S}^{-1}, \quad \operatorname{adj}\boldsymbol{S}^{-1} = |\boldsymbol{S}^{-1}|\boldsymbol{S}. \tag{3.11}$$

Therefore

$$\begin{aligned} \operatorname{adj}(\eta\boldsymbol{E} + \eta\boldsymbol{D} + \boldsymbol{F}) &= \eta^{\frac{1}{2}(n-1)}\boldsymbol{S}(\operatorname{adj}(\boldsymbol{E} + \eta^{1/2}\boldsymbol{D} + \boldsymbol{F}))\boldsymbol{S}^{-1} \\ &= \eta^{\frac{1}{2}(n-m)}\boldsymbol{S}(\operatorname{adj}(\boldsymbol{E} + \eta^{1/2}\boldsymbol{D} + \boldsymbol{F}))\boldsymbol{T} \end{aligned} \tag{3.12}$$

where

$$\boldsymbol{T} = \begin{bmatrix} \eta^{\frac{1}{2}(m-1)}\boldsymbol{I}_1 & & & & & & & \\ & \cdot & & & & & & \\ & & \cdot & & & & & \\ & & & \cdot & & & & \\ & & & & \cdot & & & \\ & & & & & \cdot & & \\ & & & & & & \eta^{1/2}\boldsymbol{I}_{m-1} & \\ & & & & & & & \boldsymbol{I}_m \end{bmatrix}. \tag{3.13}$$

Each of the three matrices $\boldsymbol{S}$, $\operatorname{adj}(\boldsymbol{E} + \eta^{1/2}\boldsymbol{D} + \boldsymbol{F})$ and $\boldsymbol{T}$ contains only non-negative powers of $\eta^{1/2}$. Thus, $\frac{1}{2}(n-m)$ is a common factor of all elements of $\operatorname{adj}(\eta\boldsymbol{E} + \eta\boldsymbol{D} + \boldsymbol{F})$, and hence it is a factor of $D(\eta)$. We shall investigate the problem of whether any additional power of $\eta^{1/2}$ can appear as a common factor of the elements of $\operatorname{adj}(\boldsymbol{E} + \eta^{1/2}\boldsymbol{D} + \boldsymbol{F})$.

For brevity we shall write

$$\lambda = \eta^{1/2}. \tag{3.14}$$

The elements of $\operatorname{adj}(\boldsymbol{E} + \lambda\boldsymbol{D} + \boldsymbol{F})$ are polynomials in λ of degree $(n-1)$ or less. They will have a common factor λ if and only if all elements of $\operatorname{adj}(\boldsymbol{E} + \lambda\boldsymbol{D} + \boldsymbol{F})$ are zero when $\lambda = 0$. But the matrix $(\boldsymbol{E} + \boldsymbol{F})$ has rank $(n-k)$ (cf. Tee (1963), equation (3.4)). Therefore, if

$k = 0$ or 1, at least one minor of order $(n - 1)$ must exist which is non-zero when $\lambda = 0$. Hence, λ cannot be a common factor of the elements of adj$(\boldsymbol{E} + \lambda\boldsymbol{D} + \boldsymbol{F})$ if $k = 0$ or $k = 1$.

In general,

$$\text{adj}(\boldsymbol{E} + \lambda\boldsymbol{D} + \boldsymbol{F}) = |\boldsymbol{E} + \lambda\boldsymbol{D} + \boldsymbol{F}|(\boldsymbol{E} + \lambda\boldsymbol{D} + \boldsymbol{F})^{-1}$$

$$= \lambda^k Q(\lambda^2)(\boldsymbol{E} + \lambda\boldsymbol{D} + \boldsymbol{F})^{-1}. \quad (3.15)$$

(cf. Tee (1963), equation (3.3)) where $Q(0) \neq 0$. Therefore,

$$\lambda^{-k}\text{adj}(\boldsymbol{E} + \lambda\boldsymbol{D} + \boldsymbol{F}) = Q(\lambda^2)(\boldsymbol{E} + \lambda\boldsymbol{D} + \boldsymbol{F})^{-1}. \quad (3.16)$$

Hence, if λ^k were a common factor of the elements of adj$(\boldsymbol{E} + \lambda\boldsymbol{D} + \boldsymbol{F})$, then the right-hand side of (3.16) would be finite when $\lambda = 0$, i.e. $Q(0)(\boldsymbol{E} + \boldsymbol{F})^{-1}$ would exist. But $Q(0) \neq 0$ and therefore $(\boldsymbol{E} + \boldsymbol{F})^{-1}$ would exist, which is impossible if $k \neq 0$. Therefore, λ^k cannot be a common factor of the elements of adj$(\boldsymbol{E} + \lambda\boldsymbol{D} + \boldsymbol{F})$ if $k \neq 0$.

Any element of adj$(\boldsymbol{E} + \lambda\boldsymbol{D} + \boldsymbol{F})$ is an $(n - 1)$th order cofactor of $(\boldsymbol{E} + \lambda\boldsymbol{D} + \boldsymbol{F})$. Since one and only one term in λ appears in each row and column of $(\boldsymbol{E} + \lambda\boldsymbol{D} + \boldsymbol{F})$, the coefficient C_s of any term $C_s\lambda^s$ in the expansion of any $(n - 1)$th order cofactor will be a sum of $(n - 1 - s)$th order minors of $(\boldsymbol{E} + \boldsymbol{F})$—each minor being taken with the appropriate sign. But all minors of order greater than $n - k$ are zero, since the rank of $(\boldsymbol{E} + \boldsymbol{F})$ is $n - k$. Therefore $C_s = 0$ if $n - 1 - s > n - 1$, i.e. if $s < k - 1$ (and hence $k > 1$). Hence, if $k > 1$, a factor of λ^{k-1} can be extracted from each element of adj$(\boldsymbol{E} + \lambda\boldsymbol{D} + \boldsymbol{F})$. But it was shown in the previous paragraph that (if $k > 0$)λ^k is not a common factor of the elements of adj$(\boldsymbol{E} + \lambda\boldsymbol{D} + \boldsymbol{F})$, so that λ^{k-1} must be the highest common power of λ.

Summarizing, the highest power of λ which is a common factor of the elements of adj$(\boldsymbol{E} + \lambda\boldsymbol{D} + \boldsymbol{F})$ is λ^ν, where

$$\left.\begin{array}{ll} \nu = 0 & \text{if } k = 0 \\ \nu = k - 1 & \text{if } k > 0 \end{array}\right\}. \quad (3.17)$$

Therefore, the highest power of λ which is a common factor of the elements of $\boldsymbol{S}(\text{adj}(\boldsymbol{E} + \lambda\boldsymbol{D} + \boldsymbol{F}))\boldsymbol{T}$ may be written as $\lambda^{\nu+a}$, where $a \geqslant 0$. It follows from (3.12) that the highest power of η which is a common factor of the elements of adj$(\eta\boldsymbol{E} + \eta\boldsymbol{D} + \boldsymbol{F})$ is

$$D(\eta) = \eta^{1/2(n-m+\nu+a)}. \quad (3.18)$$

Substituting (3.18) and (3.3) into (3.1), we get

$$M(\eta) = \frac{T(\eta)}{D(\eta)} = \frac{\eta^{\frac{1}{2}(n+k)}\,Q(\eta)}{\eta^{\frac{1}{2}(n-m+\nu+a)}} = \eta^{\frac{1}{2}(m+k-\nu-a)}Q(\eta) \quad (3.19)$$

where $Q(0) \neq 0$.

Upper bound for m_1

The theorem cited at the beginning of § 3 shows that

$$m_1 = \tfrac{1}{2}(m + k - \nu - a) = \tfrac{1}{2}(p + q - 3 + k - \nu - a). \quad (3.20)$$

Substituting (3.17) into (3.20), we see that if p and q are co-prime so that $k = 0$ (cf. (2.11)), then

$$m_1 = \tfrac{1}{2}(p + q - 3 - a) \leqslant \tfrac{1}{2}(p + q - 3), \quad (3.21)$$

but if $k \neq 0$ then

$$m_1 = \tfrac{1}{2}(p + q - 2 - a) \leqslant \tfrac{1}{2}(p + q - 2). \quad (3.22)$$

Moreover, $p \leqslant q$ so that if $k = 0$ we have

$$m_1 \leqslant \tfrac{1}{2}(2q - 3) = q - \tfrac{3}{2}$$

or, since m_1 is an integer,

$$m_1 \leqslant q - 2; \quad (3.23)$$

and if $k \neq 0$ then

$$m_1 \leqslant q - 1. \quad (3.24)$$

4. Conclusions

Combining (3.21) — (3.24) with (2.13), we see that if $k = 0$ then

$$\tfrac{1}{2}(q - 1) \leqslant m_1 \leqslant \tfrac{1}{2}(p + q - 3) \leqslant q - 2 < q - 1, \quad (4.1)$$

and if $k \neq 0$ then

$$\tfrac{1}{2}(q - 1) \leqslant m_1 \leqslant \tfrac{1}{2}(p + q - 2) \leqslant q - 1 \quad (4.2)$$

where $p \leqslant q$.

Define r as the number of rows or columns (whichever is the greater) of internal nodes in the rectangular net. Then we have shown that the number m_1 of iterations of the Liebmann process, which are needed to annihilate all components of the initial error corresponding to zero eigenvalue, satisfies the inequality

$$\tfrac{1}{2}r \leqslant m_1 \leqslant r. \quad (4.3)$$

Only after this number of iterations can the current error be expressed in terms of eigenvectors alone of $\boldsymbol{H}$.

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