A study of the solution of an initial-value problem with a hybrid computer

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In the solution of problems involving partial differential equations the analogue computer has for a long time suffered a disadvantage: namely, the lack of a large store and of logical devices, which would enable iterative techniques to be used. This state of affairs is now being rectified by several analogue computer manufacturers in varying degrees, their products being called *hybrid* computers.

This paper looks into the possibility of the solution on a hybrid of a particular type of initial-value problem involving partial differential equations, that has for some time been studied at the C.E.G.B. Analogue Computation Centre at Friars House: namely, reactor fault studies on a single channel in the flattened zone of a nuclear reactor. So far the difference-methods employed have of necessity involved a coarse mesh; with the hybrid the mesh can be made finer, thus presenting a better approximation. The paper then poses a simplified problem, and the errors involved in two methods of solution that could be used on the hybrid computer are assessed and compared with those of the corresponding methods that would be used on the digital computer.

1. Brief description of the hybrid computer considered

The hybrid computer considered is an ordinary modern analogue computer fitted with the following additional equipment:

(1) equipment to permit high-speed repetitive operation
(2) digital/analogue converters
(3) logical devices
(4) stores.

The logical devices are primarily to be used to enable decisions concerning the mode of integrators of the analogue computer to be taken and implemented in the course of the solution of a problem. This, together with the stores, enables rapid iterative techniques to be used, as in the subject of the study described later. Although not considered here, arithmetic units could also be added to the above list of additional equipment if a given problem required the use of these units in its solution. Computers with these additional items of equipment are now commercially available.

It is shown in the following Sections, by means of an example, that it is possible to solve an initial-value problem on an hybrid assembly, such as is described above. Schemes are evolved belonging to the categories *continuous space variable*, *discrete time* to be labelled CSDT and *discrete space variable*, *continuous time*, to be labelled DSCT.

Any scheme used on a digital computer belongs to the category *discrete space variable*, *discrete time* DSDT. Richtmyer in his book (1957) gives many schemes of DSDT for the solution of various problems, and in particular for the diffusion problem, and analyzes their performance. His methods of analysis are modified in this paper to be appropriate to the schemes evolved therein when applied to the diffusion equation. Thereby comparisons between similar schemes in CSDT and DSDT, and in DSCT and DSDT can be made. This comparison is presented in Table 1.

It will be seen that from the point of view of accuracies the various schemes have near equal merit. There is, however, a distinct advantage of the two categories CSDT and DSCT over DSDT applied to the hybrid and digital computers, respectively, in that the number of computations of any scheme of the former is much less than any of the latter. Whereas in the digital computer the integration with respect to both independent variables is performed by the summation of a large number of quantities successively, in the hybrid computer the integration with respect to one independent variable is continuously carried out by the charging or discharging of a capacitor. This is one simple operation, and it can be made as rapid as one may wish, subject of course, to an upper limit determined by the characteristics of the amplifier, which in currently commercially available computers with high-speed repetitive operation is sufficiently high to make the method feasible.

2. Problem to be solved

The temperatures $T_i \equiv T_i(z, t)$ $(i = 1, 2, \ldots, \mathrm{I})$, and the neutron flux $\phi(z, t)$ over the region $0 \leqslant z \leqslant h$, $0 \leqslant t \leqslant \tau$, z the distance along it, are sought for a single channel in the flattened region of a reactor disturbed from a steady state. The equations connecting these variables are of the form

Thermal

$$C_i \frac{\partial T_i}{\partial t} = \delta_{ic} G \frac{\partial T_i}{\partial z} + \sum_{\substack{k \\ k \neq c}} \delta_{ik} E_i \phi + \sum_{j=1}^{I} Q_j (T_i - T_j) \quad (2a)$$

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Table 1

Comparisons of difference schemes

applied to $\frac{\partial u'}{\partial t} = \sigma \frac{\partial^2 u'}{\partial x^2}$

Legend: $\theta = $ const., $0 \leqslant \theta \leqslant 1$ $e = $ truncation error $p = $ greatest number for which $\partial^p u'/\partial x^p\vert_{t=0}$ exists $(\delta^2 u)_j^n = u_{j+1}^n - 2u_j^n + u_{j-1}^n$ $u_j^n = u(j\Delta x, n\Delta t)$	*DSDT* $\left(\begin{matrix} t = u\Delta t \\ x = j\Delta x\end{matrix}\right)$ $\frac{u_j^{n+1} - u_j^n}{\Delta t} = \sigma \frac{\theta(\delta_u^2)_j^{n+1} + (1-\theta)(\delta^2 u)_j^n}{\Delta x^2}$ $e = O(\Delta t) + O(\Delta x^2)$ $= O(\Delta t^2) + O(\Delta x^4)$ if $\theta = \frac{1}{2}$
CSDT $(t = n\Delta t)$ $\frac{u_{n+1} - u_n}{\Delta t} = \sigma\left[\theta \frac{d^2 u_{n+1}}{dx^2} + (1-\theta)\frac{d^2 u'_n}{dx^2}\right]$ $e = O(\Delta t)$ $= O(\Delta t^2)$ if $\theta = \frac{1}{2}$ stable if $\frac{1}{2} - \frac{1}{\sigma \Delta t} \leqslant \theta \leqslant 1$, unstable if $0 \leqslant \theta < \frac{1}{2} - \frac{1}{\sigma\Delta t}$ $u' - u = O[\Delta t^{p/(p+4)}]$ $= O[\Delta t^{2p/(p+6)}]$ if $\theta = \frac{1}{2}$	stable for $0 \leqslant \theta < \frac{1}{2}$ if $\sigma \Delta t/\Delta x^2 = $ const. $\leqslant 1/(2-4\theta)$ for $\frac{1}{2} \leqslant \theta \leqslant 1$ unconditionally $u' - u = O[\Delta t^{(p+2)/(p+4)}]$ $= O[\Delta t^{(2p+3)/(p+6)}]$ if $\sigma\Delta t/\Delta x^2 = 1/6$ (no special investigation given for $\theta = \frac{1}{2}$)
DSCT $(x = j\Delta x)$ $\theta \frac{du_{j+1}}{dt} + 2(1-\theta)\frac{du_j}{dt} + \theta\frac{du_{j-1}}{dt} = \frac{2\sigma}{\Delta x^2}(u_{j+1} - 2u_j + u_{j-1})$ $e = O(\Delta x^2)$ $= O(\Delta x^4)$ if $\theta = \frac{1}{6}$ always stable $u' - u = O(\Delta x^2)$ $= O(\Delta x^4)$ if $\theta = \frac{1}{6}$	$\frac{1}{12}\frac{u_{j+1}^{n+1} - u_{j+1}^n}{\Delta t} + \frac{5}{6}\frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{1}{12}\frac{u_{j-1}^{n+1} - u_{j-1}^n}{\Delta t} = \sigma \frac{(\delta^2 u)_j^{n+1} + (\delta^2 u)_j^n}{2\Delta x^2}$ $e = O(\Delta t^2) + O(\Delta x^4)$ always stable behaviour of $u' - u$ at $\Delta t \to 0$ not given (The information of this column is taken from Richtmyer (1957))

where

$$\delta_{ij} = \begin{cases} 1, i = j \text{ and } & i = 1, 2, \ldots, I \\ 0, i \neq j \text{ (some of the } Q_j \text{ depend on } T_i \text{ and on } G). \end{cases}$$

Neutron flux

$$M^2 \frac{\partial^2 \phi}{\partial z^2} + K(z, t, T_i)\phi + \sum_i \mu_i P_i = l \frac{\partial \phi}{\partial t}. \qquad (2b)$$

Delayed neutrons

$$\frac{\partial P_i}{\partial t} = \mu_i P_i + \alpha_i \phi. \qquad (2c)$$

Control $K(z, t, T_i) = K'(z, t, T_i) + a\, s(t)$ (2*d*)

$$\frac{ds}{dt} = f(T_c(h) - \beta).$$

Trips $K'(z, t, T_i) = K''(z, T_i) + \sum_j b_j s_j(t)$ (2*e*)

$$\frac{ds_j}{dt} = \begin{cases} 0 & 0 \leqslant t < \tau \\ \alpha_j & \tau \leqslant t. \end{cases}$$

Boundary conditions

$$\left.\begin{array}{ll} \text{(i)} & T_c(0) = \text{const} \\ \text{(ii)} & \phi(0, t) = d_1 \left.\frac{\partial \phi}{\partial z}\right|(0^+, t) \\ \text{(iii)} & \phi(h, t) = -d_2 \left.\frac{\partial \phi}{\partial z}\right|(h^-, t) \\ \text{(iv)} & \phi \geqslant 0, 0 \leqslant z \leqslant h. \end{array}\right\} 0 \leqslant t \leqslant \tau \quad (2f)$$

In the autonomous system (steady state) $T_c(h) = \beta$.

Initial conditions (2*g*)

$T_i(z, 0)$, $\phi(z, 0)$ steady-state values.

$s_i(0)$ are given.

Fault $G = \Gamma(t)$, a given function on $[0, \tau]$. (2*h*)

3. Methods of solution on the hybrid computer

3.1. *Continuous space variable* (z), *discrete time* (CSDT)

Each of the differential equations of Section 2 is of the form

$$\frac{\partial X}{\partial t} = F(z, t)$$

and we are going to replace t by $n\Delta t$, $n = 0, 1, 2, \ldots, N$. By the mean-value theorem

$$X(z, t + \Delta t) - X(z, t) = \Delta t \left.\frac{\partial X}{\partial t}\right|_{(z, t+\mu\Delta t)}, \quad 0 < \mu < 1$$
$$= \Delta t . F(z, t + \mu\Delta t). \qquad (3.1a)$$

Here the inevitable approximation is made: one way of doing this is to replace $F(z, t + \mu\Delta t)$ by a mean value over the interval $(t, t + \Delta t)$, in fact, by

$$(1 - \theta)F(z, t) + \theta F(z, t + \Delta t)$$

where $0 \leqslant \theta \leqslant 1$ is a constant, in general differing with each equation, whose value will have to be decided on by consideration of stability and the truncation error (see Section 4). Hence replacing t by $n\Delta t$ and making an obvious change in notation we have

$$X_{n+1}(z) - X_n(z) = \Delta t[(1-\theta)F_n(z) + \theta F_{n+1}(z)],$$
$$n = 0, 1, 2, \ldots, N; \quad 0 \leqslant \theta \leqslant 1. \quad (3.1b)$$

Thus at any stage $(n + 1)$ of the sequence of computations, the analogue computer solves for $X_{n+1}(z)$ on the interval $0 \leqslant z \leqslant h$, X_n and F_n being supplied to it from stores, having been put in to these during the previous stage n. It must be remarked here that it is only a function sampled at a finite number of points that is stored. However, the number of points available in stores of one manufacturer is large, in fact 256, and then a circuit can be arranged for linear or other interpolation.

In this problem X_0 the solution of the autonomous system is not known and has to be found by the computer. Equations (2*b*) and (2*f*) in their autonomous form show that this is an eigenvalue problem. The numerical values of the data are generally inaccurate and so an adjustment has to be made to obtain a solution satisfying (2*f*). The term k_∞ contained in K in (2*b*) is multiplied by a number λ_ν. Iterating the computer with a suitable circuit generates a sequence (λ_ν) converging to λ, the lowest eigenvalue, and thus X_0 is obtained. Details of such a procedure have been worked out by Vichnevetsky (1962).

Once X_0 is determined the computer moves on to stage $n = 1$. In solving for X_1 it must iterate until condition (2*f*) (iii) is satisfied, the parameter of variation being the initial condition, $\phi_1(0)$. So the computer determines a sequence (v_ν) converging to v, the correct value to be assumed by $\phi_1(0)$. Thus is X_1 obtained, and so on. The iteration at each stage n has, of course, a finite number of steps, it being automatically stopped when a predetermined error is achieved.

Another possible scheme is obtained from (3.1*a*) together with

$$X(z, t - \Delta t) - X(z, t) = -\Delta t F(z, t - \mu'\Delta t),$$
$$0 < \mu' < 1.$$

Adding $(1 + \alpha)$ times the first to $(-\alpha)$ times the second, where $\alpha \geqslant 0$, gives

$$(1+\alpha)[X(z, t+\Delta t) - X(z, t)] - \alpha[X(z, t) - X(z, t-\Delta t)]$$
$$= \Delta t\{F(z, t+\mu\Delta t) + \alpha[F(z, t+\mu\Delta t) - F(z, t-\mu'\Delta t)]\}$$
$$= \Delta t\, F(z, t + \mu\Delta t) + 0(\Delta t^2) \quad \text{as } \Delta t \to 0$$
$$\sim \Delta t F(z, t + \mu\Delta t).$$

Then replacing $F(z, t + \mu\Delta t)$ by the mean value as before, we get

$$(1 + \alpha)(X_{n+1} - X_n) - \alpha(X_n - X_{n-1})$$
$$= \Delta t[(1 - \theta)F_n + \theta F_{n+1}],$$
$$n = 0, 1, 2, \ldots, N$$
$$0 \leqslant \theta \leqslant 1$$
$$\alpha \geqslant 0.$$

This method is not pursued further in this paper.

Care should be taken that Δt is sufficiently small to avoid spurious solutions; see, for example, Section 4.12.

3.2. *Discrete space variable, continuous time* (*DSCT*)

Here we have only two differential operators to replace by difference operators, namely $\partial T_c/\partial z$ and $\partial^2\phi/\partial z^2$ in equations (2*a*) and (2*b*). Hence we may write these equations as

$$\frac{\partial T_c}{\partial z} = F(z, t) \qquad 0 < z < h, \ t \geqslant 0$$
$$\frac{\partial^2 \phi}{\partial z^2} = G(z, t) \qquad 0 < z < h, \ t \geqslant 0,$$

and putting $z = j\Delta z$ and T and T_c replace them by

$$T_{j+1}(t) - T_j(t) = \Delta z[(1 - \theta)F_j(t) + \theta F_{j+1}(t)]$$
$$j = 1, 2 \ldots, J - 1 \quad (3.2a)$$

exactly as in 3.1, and

$$\phi_{j+1}(t) - 2\phi_j(t) + \phi_{j-1}(t)$$
$$= \frac{\Delta z^2}{2}[\theta^+ G_{j+1}(t) + (2 - \theta^+ - \theta^-)G_j(t) + \theta^- G_{j-1}(t)],$$
$$j = 2, 3, \ldots J - 1, \quad (3.2b)$$

where $0 \leqslant \theta^+ \leqslant 1$, $0 \leqslant \theta^- \leqslant 1$, which is obtained from

$$\phi(z + \Delta z, t) - \phi(z, t) =$$
$$\Delta z \left.\frac{\partial \phi}{\partial z}\right|_{(z, t)} + \frac{\Delta z^2}{\boxed{2}} \left.\frac{\partial^2 \phi}{\partial z^2}\right|_{(z+\mu\Delta z, t)}, \quad (0 < \mu < 1)$$

by the method in Section 3.1. Again values that should be given to the constants θ^+, θ^- depend on considerations of stability and truncation error. $T_1(t)$ and $\phi_2(t)$ are computed from formulae (3.2*a*) and (3.2*b*), respectively, with $\theta = 1$ and $\theta^- = 0$ in order to avoid the necessity of computing $F_0(t)$ and $G_0(t)$, for a reason that will appear later.

We have also to approximate to the ϕ-derivatives in (2*f*):

$$\phi(\Delta z, t) - \phi(0, t) = \Delta z \frac{\partial \phi}{\partial z}\bigg|_{(0^+, t)} + O(\Delta z^2) \text{ as } \Delta z \to 0, \quad \Delta z > 0$$

$$\phi(\Delta z, t) = \phi(0, t)\left(1 + \frac{\Delta z}{d_1}\right) + O(\Delta z^2) \quad \text{by } (2f).$$

So we put

$$\phi_1(t) = \phi_0(t)\left(1 + \frac{\Delta z}{d_1}\right) \tag{3.2c}$$

and similarly

$$\phi_{J-1}(t) = \phi_J(t)\left(1 + \frac{\Delta z}{d_2}\right). \tag{3.2d}$$

To obtain the steady state the computer, by an iterative procedure, seeks the eigenvalue λ so that equation (3.2*d*) is satisfied and T_J assumes the value given, the integrators of the analogue section being in a non-operate mode. Now revert to the general dependent variable X of Section 2.1, but giving a different significance to the suffix, i.e. put

$$X_j \equiv X_j(t) = X(j\Delta z, t).$$

Then, when $X_j(0)$ $(j = 0, 1, \ldots, J)$ are found, $X_j(t)$ will be solved by integration from 0 to τ for each $j = 1, 2, \ldots, J$ successively, in accordance with the relationships (3.2*a*, *b*, *c*), the stores taking in the functions $X_j, \ldots, G_{j-1}, \ldots$ over the interval $0 \leqslant t \leqslant \tau$. But, to start this, $\phi_0(t)$ must be known. Let a function $\Phi(t)$, $0 \leqslant t \leqslant \tau$, be given—the origin of this will be explained later—and put $\phi_0(t) = \Phi(t)$. The computation is then proceeded with and a comparison made between the graph $\phi_J(t)$, $0 \leqslant t \leqslant \tau$, as computed according to equation (3.2*b*) and $\phi_J^*(t)$, say, computed by (3.2*d*). A modification is then made to $\phi_0(t)$ and the process repeated. By iteration the computer determines a sequence of functions $\phi_J^{(\nu)}(t)$, $0 \leqslant t \leqslant \tau, \nu = 1, 2, \ldots$, and the process is stopped at that stage N for which

$$\max_{0 \leqslant t \leqslant \tau} |\phi_J^{(N)}(t) - \phi_J^{*(N)}(t)|$$

is less than a predetermined value.

Now we consider the origin of Φ. At the point $j = 0$ we are not interested in any of the variables except $\phi_0(t)$, for save for this the variables do not play any part in the subsequent computation. However, if we ascribe values to

$$\frac{\partial T_c}{\partial z}\bigg|_{(0, t)}, \frac{\partial^2 \phi}{\partial z^2}\bigg|_{(0, t)}, \quad 0 \leqslant t \leqslant \tau$$

we may solve the system of equations at the point $j = 0$ for $\phi_0(t)$ $0 \leqslant t \leqslant \tau$. So Φ may be defined as the solution when

$$\left.\begin{aligned} \frac{\partial T_c}{\partial z}\bigg|_{(0, t)} &= 0 \\ \frac{\partial^2 \phi}{\partial z^2}\bigg|_{(0, t)} &= 0 \end{aligned}\right\} 0 \leqslant t \leqslant \tau.$$

It would seem that the method of Vichnevetsky (1962) could be extended to apply to the iteration process of finding ϕ_0 and ϕ_J. At the commencement of stage ν of the iteration a store would feed the computer with $\phi_0^{(\nu)}(t_i)$, $0 = t_1 < t_2 < \ldots < t_M = \tau$ and the computation would proceed. Let A_ν be the set of points t_i at which $|\phi_J^{(\nu)}(t_i) - \phi_J^{*(\nu)}(t_i)| > \varepsilon$, where ε is the predetermined value of the error. Then into a second store would be written the function $y^{(\nu)}$ given by

$$y^{(\nu)}(t_i) = \begin{cases} \Delta\phi_0^{(\nu)}(t_i) & \text{if } t_i \,\varepsilon A_\nu \\ 0 & \text{otherwise} \end{cases}$$

where $|\Delta\phi_0^{(\nu)}(t_i)|$ is some function of $|\phi_J^{(\nu)}(t_i) - \phi_J^{(\nu)}(t_i)|$. The outputs of the stores would then be added to give $\phi_0^{(\nu+1)}(t_i)$, and the computation of stage $\nu + 1$ started, and so on.

4. Performance of methods

In attempting to assess the stability of a difference scheme representing the system of equations of Section 2, one is immediately confronted with the difficulties associated with the non-linearity of some of them. It is true that the coefficients of the thermal equations (2*a*), which are generally functions of a temperature, are slowly varying functions and that their variability hardly affects the behaviour of the system. But the offender, $K(z, t, T_i)$, in the neutron diffusion equation (2*b*) cannot be made independent of temperature without the system being radically altered; because it is only through this coefficient that the temperatures affect the neutron flux. Nevertheless, useful information is gained, if the stability of the linearized system is studied, even if it is assumed that the coefficients are constant.

Consider the system of Section 2 linearized; that is, consider the equation

$$\frac{\partial \boldsymbol{X}}{\partial t} = A\boldsymbol{X} \tag{4a}$$

where $\boldsymbol{X}$ is vector whose components are the dependent variables of the system, of number say p, and A is a linear operator.

4.1. *Stability*

4.11. *CSDT*.—According to Section 3.1 we would replace (4*a*) by

$$\frac{1}{\Delta t}(\boldsymbol{X}_{n+1} - \boldsymbol{X}_n) = \delta_{ij}(1 - \theta_i)A\boldsymbol{X}_n + (\delta_{ij}\theta_i)A\boldsymbol{X}_{n+1} \tag{4.11a}$$

where $(\delta_{ij}\alpha_i)$ is $p \times p$ diagonal matrix whose diagonal entries are $\alpha_1, \ldots, \alpha_p$. One could then, perhaps, follow Richtmyer (1957) and determine values of the θ_i such that the system (3.1a) was stable.

Since this is a preliminary study this course will not be followed, but instead the stability, etc., of the difference scheme representing one equation, namely (2b), will be investigated in the manner of Richtmyer, so that comparison can be made with his results.

4.12 *Simplified problem*

$\phi = \phi(z, t), \quad 0 \leqslant z \leqslant h, \quad t \geqslant 0$ is required, where

$$\frac{\partial^2\phi}{\partial z^2} + K\phi = l\frac{\partial\phi}{\partial t}, \quad t > 0; \quad K, l \text{ constants} \quad K \geqslant 0 \tag{4.12a}$$

Initial condition: $\phi(z, 0) = \Phi(z), \quad 0 \leqslant z \leqslant h \quad$ (4.12b)
where $\Phi(z)$ is a given function whose Fourier series is absolutely convergent
and $\qquad \Phi(0) = \Phi(h) = 0.$

Boundary condition:

$$\phi(0, t) = \phi(h, t) = 0, t > 0. \tag{4.12c}$$

Solution is

$$\phi = \sum_{m=1}^{\infty} b_m \sin\left(\frac{m\pi}{h}z\right) e^{-\left(\frac{m^2\pi^2}{h^2} - K\right)\frac{1}{l}t}, \quad 0 \leqslant z \leqslant h, t \geqslant 0 \tag{4.12d}$$

where

$$b_m = \frac{2}{h}\int_0^h \Phi(z) \sin\left(\frac{m\pi z}{h}\right) dz$$

if we define $\qquad -\Phi(-z) = \Phi(z)$

4.13. With the first difference scheme of 3.1 this becomes

$$l.\frac{\phi_{n+1} - \phi_n}{\Delta t} = (1 - \theta)\left(\frac{d^2\phi_n}{dz^2} + K\phi_n\right) + \theta\left(\frac{d^2\phi_{n+1}}{dz^2} + K\phi_{n+1}\right), \quad n = 0, 1, \ldots, N. \tag{4.13a}$$

Here we assume that the ϕ_n are stored continuously, a reasonable assumption in view of the remarks in Section 3.1.

Initial condition: $\qquad \phi_0(z) = \Phi(z) \qquad$ (4.13b)

where we may and do suppose that $d^3\Phi/dz^3$ exists and is of bounded variation in $[0, h]$. For if it did not exist at any point, as can happen in a problem, we could select a function Ψ, for which $d^3\Psi/dz^3$ exists everywhere, as close to Φ as we please.

Boundary conditions:

$$\phi_n(0) = \phi_n(h) = 0, \quad n = 0, 1, \ldots N. \tag{4.13c}$$

Note that in (4.12a) and (4.12b) the case $t = 0$ is excluded while in (4.13a) and (4.13b) the case $n = 0$ is necessarily included and hence the restriction on $\Phi(z)$ in (4.13b).

If $\Delta t \leqslant l/\theta K$ and $0 < \theta \leqslant 1$ the solution is

$$\phi_n = \sum_{m=1}^{\infty} b_m \sin\left(\frac{m\pi}{h}z\right)\xi_m^n \quad (n = 0, 1, 2, \ldots N) \tag{4.13d}$$

where b_m is as defined in (4.12b) and

$$\xi_m \equiv \xi\left(\frac{\Delta t}{l}k_m\right) = \frac{1 - (1 - \theta)\frac{\Delta t}{l}k_m}{1 + \theta\frac{\Delta t}{l}k_m}$$

$$k_m = \frac{m^2\pi^2}{h^2} - K, \quad m = 1, 2, \ldots.$$

For $b_m \sin\frac{m\pi}{h}z . \xi_m^n$ satisfies (4.13a) for each n.

From the hypothesis in (4.13b) it can be shown that $b_m = O\left(\frac{1}{m^4}\right)$ as $m \to \infty$ and so

$$\left|\frac{d^2}{dz^2}\sum_{m=1}^{\infty} b_m \sin\left(\frac{m\pi}{h}z\right)\xi_m^n\right| = \left|\sum_{m=1}^{\infty} - m^2 b_m \sin\left(\frac{m\pi}{h}z\right)\xi_m^n\right| \leqslant C\sum_{m=1}^{\infty}\frac{1}{m^2} < +\infty$$

since $\xi_m \sim -\frac{1}{\theta} + 1$ as $m \to \infty$*. Hence (4.13d) satisfies (4.13a). Clearly it satisfies (4.13b) and (4.13c).

The restriction $\Delta t \leqslant \frac{l}{\theta K}$ ensures that the solution of the homogeneous equation

$$\frac{d^2\phi_{n+1}}{dz^2} + \left(K - \frac{l}{\theta\Delta t}\right)\phi_{n+1} = 0$$

that satisfies (4.13c) is $\phi_{n+1} \equiv 0$. So (4.13d) is the unique solution of the difference-method form of the problem.

ξ_m is the amplification factor, as Richtmyer calls it. It is analogous to the factor

$$e^{-k_m t/l}$$

in (4.12d). The necessary and sufficient condition for stability (Richtmyer, 1957) of the difference scheme is that there should be an $\eta > 0$ such that

$$|\xi_m| \leqslant 1 + O(\Delta t), \quad 0 < \Delta t < \eta \quad m = 1, 2, \ldots.$$

Now (k_m) is a strictly increasing unbounded sequence, so there is a least number m_0 such that $k_{m_0} \geqslant 0$, whatever K may be. So

$$|\xi_m| \leqslant 1 \quad m \geqslant m_0$$
$$|\xi_m| > 1 \quad 1 \leqslant m < m_0 \quad (\text{if } m_0 > 1).$$

* If $\theta = 1$, then $\xi_m = O\left(\frac{1}{m^2}\right)$ as $m \to \infty$ and then the restriction of (4.13b) could be relaxed to the necessity of the existence of $\frac{d\Psi}{dz}$ in [0, h].

For $m \geqslant m_0$

$$-1 \leqslant \frac{1-(1-\theta)\Delta t k_m/l}{1+\theta\Delta t k_m/l} \leqslant 1$$

if and only if

$$\theta \geqslant \frac{1}{2} - \frac{1}{\Delta t k_m} \geqslant \frac{1}{2} - \frac{1}{\Delta t k_{m_0}}$$

while for $1 \leqslant m < m_0$

$$\frac{1-(1-\theta)\Delta t k_m/l}{1+\theta\Delta t k_m/l} = 1 + \Delta t|k_m|/l + 0(\Delta t^2) \text{ as } \Delta t \to 0.$$

Hence the system is stable if and only if

$$\frac{1}{2} - \frac{1}{\Delta t k_{m_0}} \leqslant \theta.$$

If Φ is such that there is only a finite number of its Fourier coefficients different from zero, then the system will be obviously stable for any value of θ. However, due to rounding and other errors it is impossible to present the computer with such a Φ exactly; hence the necessity of the condition.

4.14. *DSCT.*—As in the case of CSDT we shall consider only the stability of the difference scheme appropriate to this category applied to the simplified problem of Section 4.12.

This becomes, with the difference scheme of Section 3.2,

$$\phi_{j+1} - 2\phi_j + \phi_{j-1} = \frac{\Delta z^2}{2}\left(l\frac{d}{dt} - K\right)(\theta\phi_{j+1} + 2(1-\theta)\phi_j + \theta\phi_{j-1}),$$
$$j = 1, 2, \ldots, J, \quad (4.14a)$$

where we have put $\theta^+ = \theta^-$, together with conditions (4.12*b*) and (4.12*c*) with z replaced by $j\Delta z$. $(j = 1, 2, \ldots J)$

The solution of this is

$$\phi_j = \sum_{m=1}^{\infty} b_m \sin m\frac{\pi}{h}j\Delta z . e^{-x_m t/l}$$

where $$x_m = \frac{2}{\Delta z^2}\,\frac{1-\cos m\frac{\pi}{h}\Delta z}{1-\theta\left(1-\cos m\frac{\pi}{h}\Delta z\right)} - K, \theta \neq 1.$$

Since $x_m = k_m + 0(\Delta z^2)$ for each m as $\Delta z \to 0$ there is no problem of stability.

4.2. *Truncation error*

4.21. *With CSDT.*—This is given by

$$e[\phi] = l\frac{\phi_{n+1}-\phi_n}{\Delta t} - (1-\theta)\left(\frac{d^2\phi_n}{dz^2} + K\phi_n\right) - \theta\left(\frac{d^2\phi_{n+1}}{dz^2} + K\phi_{n+1}\right) - \left(l\frac{\partial\phi}{\partial t} - \frac{\partial^2\phi}{\partial z^2} - K\phi\right)_n$$

and by means of the mean-value theorem one sees that

$$e[\phi] = \frac{\Delta t}{2}\frac{\partial}{\partial t}\left[l\frac{\partial\phi}{\partial t} - 2\theta\left(\frac{\partial^2\phi}{\partial z^2} + K\phi\right)\right]\bigg|_n + \mathrm{O}(\Delta z^2) \text{ as}$$

$\Delta z \to 0$, it being assumed that the derivatives exist.

If u is the solution of $l\frac{\partial\phi}{\partial t} = \frac{\partial^2\phi}{\partial z^2} + K\phi = 0$

then

$$e[u] = l\frac{u_{n+1}-u_n}{\Delta t} - (1-\theta)\left(\frac{d^2u_n}{dz^2} + Ku_n\right) - \theta\left(\frac{d^2u_{n+1}}{dz^2} + Ku_{n+1}\right)$$

$$= \mathrm{O}(\Delta t) \quad \text{as } \Delta t \to 0, \text{ and in particular, if } \theta = \tfrac{1}{2}$$

$$e[u] = \mathrm{O}(\Delta t^2).$$

In either case $e[u] = \mathrm{o}(1)$ as $\Delta t \to 0$ and so (Richtmyer, 1957) the time difference scheme (4.13*a*) represents a *consistent approximation* of the initial-value problem of Section 4.12.

4.22. *With DSCT.*—Here we have

$$e[\phi] = \frac{\phi_{j+1} - 2\phi_j + \phi_j}{\Delta z^2} + \frac{1}{2}\left(K - l\frac{d}{dt}\right)(\theta^+\phi_{j+1} + (2-\theta^+-\theta^-)\phi_j + \theta^-\phi_{j-1}) - \left(\frac{\partial^2\phi}{\partial z^2} + K\phi - \frac{\partial\phi}{\partial t}\right)_j$$

and by repeated application of the mean-value theorem, it being assumed ϕ is "sufficiently" differentiable,

$$e[\phi] = \frac{1}{2}\Delta z\left(K - l\frac{\partial}{\partial t}\right)(\theta^+ - \theta^-)\frac{\partial\phi}{\partial z}\bigg|_j + \mathrm{O}(\Delta z^2) \quad \text{as } \Delta z \to 0.$$

An obvious gain is achieved if we put $\theta^+ = \theta^- = \theta$. Then

$$e[\phi] = \frac{1}{12}\Delta z^2\frac{\partial^2}{\partial z^2}\left[\frac{\partial^2\phi}{\partial z^2} + 6\theta\left(K\phi - l\frac{d\phi}{\partial t}\right)\right]\bigg|_j + \mathrm{O}(\Delta z^4).$$

If u is as in Section 4.21, then

$$e[u] = \frac{u_{j+1} - 2u_j + u_{j-1}}{\Delta z^2} + \frac{1}{2}\left(K - l\frac{d}{dt}\right)(\theta\phi_{j+1} + 2(1-\theta)\phi_j + \theta\phi_{j-1})$$

$$= \mathrm{O}(\Delta z^2) \quad \text{as} \quad \Delta z \to 0$$
$$= \mathrm{O}(\Delta z^4) \quad \text{if} \quad \theta = \tfrac{1}{6}.$$

4.3. *Rate of convergence*

The purpose of an estimation of this is to make a comparison of it with that for the case in which both differential operators are replaced by difference operators (DSDT) which is estimated by Richtmyer (see Table 1).

4.31. *With CSDT*

Let $\phi(z, n\Delta t)$ be the solution of (4.12*a*, *b*, *c*).

Let $\phi'(z, n\Delta t)$ be the solution of (4.13*a*, *b*, *c*).

We wish to study the behaviour of $\phi' - \phi$ as $\Delta t \to 0$. Put

$$\phi' - \phi = \left(\sum_{m=1}^{M} + \sum_{m=M+1}^{\infty}\right) b_m \sin \frac{m\pi}{h} \left[\xi_m^{t/\Delta t} - (e^{-K_m \Delta t/l})^{t/\Delta t}\right]$$

$$= \Sigma_1 + \Sigma_2 \text{ say, M being arbitrary.}$$

Fix $n = t/\Delta t$, and let ζ be complex; then

$$\xi^n(\zeta) = \left[\frac{1 - (1-\theta)\zeta}{1 + \theta\zeta}\right]^n, \quad 0 \leqslant \theta \leqslant 1$$

$$= 1 - n\zeta + \left[\frac{n(n-1)}{2} + n\theta\right]\zeta^2 - \left[\frac{n(n-1)(n-2)}{6} + n(n-1)\theta + n\theta^2\right]\zeta^3 + \ldots$$

inside its circle of convergence, which is $|\zeta| = \theta^{-1}$. Hence for $|\zeta| \leqslant \theta^{-1} - \delta, (\delta > 0)$

$$\frac{\xi^n(\zeta) - e^{-n\zeta}}{\zeta^2}$$

is an analytic function and therefore bounded.

In particular if $\theta = \frac{1}{2}$, then

$$\frac{\xi^n(\zeta) - e^{-n\zeta}}{\zeta^3}$$

is analytic and bounded in the region $|\zeta| \leqslant 2 - \delta$. While for positive real ζ, $\zeta \geqslant \delta > 0$

$$\frac{\xi^n(\zeta) - e^{-n\zeta}}{\zeta^p}, \quad p \geqslant 1$$

is obviously bounded.

$$\text{So } |\xi^{t/\Delta t}(k_m \Delta t/l) - (e^{-k_m \Delta t/l})^{t/\Delta t}| \leqslant B\frac{t}{\Delta t}(k_m \Delta t/l)^2$$

for $m = 1, 2, \ldots$, B being a constant, provided $\Delta t k_1 \leqslant l(\theta - \delta)$ if $k_1 < 0$. Therefore

$$|\Sigma_1| \leqslant \sum_{m=1}^{M} |b_m| . B \frac{\Delta t}{l^2} k_m^2 \leqslant CM^2 \Delta t \quad (C \text{ a constant}).$$

If $\theta = \frac{1}{2}$ then we may sharpen this result to

$$|\Sigma_1| \leqslant Ck_M^3 \Delta t^2.$$

On the other hand

$$|\Sigma_2| \leqslant \sum_{m=M+1}^{\infty} |b_m| \{|\zeta_m|^{t/\Delta t} + e^{-k_m t/l}\}$$

$$\leqslant D \sum_{n=M+1}^{\infty} |b_m|, \text{ where } D \text{ is a constant.}$$

If $b_m = O(1/m^{p+1})$ as $m \to \infty$ then

$$|\Sigma_2| = O(1/M^p)$$

and so

$$\phi' - \phi = O(1/M^p) + O(M^2 \Delta t).$$

Up to now M has been arbitrary. Now make M proportional to

$$(\Delta t)^{-1/(p+4)}$$

then $\qquad \phi' - \phi = O(\Delta t^{p/(p+4)})$ as $\Delta t \to 0$.

In particular if $p = 3$ as in (4.13*b*) $\phi' - \phi = O(\Delta t^{3/7})$ whereas if $\Phi(x)$ has derivatives of all orders

$$\phi' - \phi = O(\Delta t).$$

If $\theta = \frac{1}{2}$ these are improved to

$$\phi' - \phi = O(\Delta t^{2p/(p+6)})$$

and $\phi' - \phi = O(\Delta t^2)$ as $\Delta t \to 0$, respectively.

4.32 *With DSCT*.—In this case we have simply that

$$\phi' - \phi = \sum_{m=1}^{\infty} b_m \sin \frac{m\pi}{h} z(e^{-xt/l} - e^{-k_m t/l}).$$

It is easily seen that

$$x_m + K = \frac{m^2\pi^2}{h^2}\left[1 - (1 - 6\theta)\frac{m^2\pi^2}{12h^2}\Delta z^2 + O(\Delta z^4)\right]$$

as $\Delta z \to 0$. So that $x_m = k_m + O(\Delta z^2)$ and in particular if $\theta = \frac{1}{6}$, $x_m = k_m + O(\Delta z^4)$ as $\Delta z \to 0$.

It follows that $\phi' - \phi = O(\Delta z^2)$ as $\Delta z \to 0$; but if $\theta = 1/6$, then $\phi' - \phi = O(\Delta z^4)$ as $\Delta z \to 0$.

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