Axially symmetric steady motion of a viscous incompressible fluid: some numerical experiments

By M. R. Abbott*

This paper considers the numerical solution of axially symmetric steady motion of a viscous incompressible fluid in a circular cylinder of constant radius. A boundary-layer type assumption is made which reduces the Navier–Stokes equations from elliptic to parabolic and renders the solution independent of downstream data. Explicit methods of solution are shown to be unsatisfactory, and in practice the fully implicit method has advantages over the Crank–Nicolson method. Hagen–Poiseuille flow is used as a test case. An example is solved with an initial flow of arbitrary form, which correctly tends to Hagen–Poiseuille flow as the numerical solution is continued downstream.

Abbott and Hall (1962) describe a numerical method used for calculating the velocity and pressure in an axially symmetric flow of an incompressible inviscid fluid, given appropriate boundary conditions at an outer surface, the axis and some initial upstream cross-section. (The problem has an application to the cores of spiralling fluid formed by the rolling up of the shear layer that separates from the leading edge of a slender wing at incidence.) The neglect of viscosity, though reducing the order of the equations, brings in singularities on the axis, a difficult non-linear two-point boundary-value problem, and difficulties if the outer surface is a stream surface.

In this paper we consider numerical methods for the solution of the corresponding viscous problem, which is free from singularities and gives a realistic account of the motion near the axis. The boundary conditions are taken as far as possible similar to those for the inviscid case: data is required on an upstream section, on the axis and on an outer bounding surface; a boundary-layer type assumption is made which reduces the equations from elliptic to parabolic, and hence renders the solution independent of downstream data. The results are given of applying various numerical methods of solution to the equations of this problem. For simplicity the solution is sought in a cylindrical boundary of constant radius: flows in a general region can be solved by the same method after a change of independent variable.

Cylindrical polar coordinates (x, r, θ) are used with the x-axis along the axis of the cylinder and with the r-coordinate at right-angles to it. Conditions are independent of θ due to the symmetry. The velocity components are taken as u in the axial direction, v in the circumferential direction and w in the radial direction. The Navier–Stokes equations are then

$$u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial r} + \frac{1}{\rho}\frac{\partial p}{\partial x} = \nu\nabla^2 u, \qquad (1)$$

$$u\frac{\partial v}{\partial x} + w\frac{\partial v}{\partial r} + \frac{vw}{r} = \nu\left(\nabla^2 v - \frac{v}{r^2}\right), \qquad (2)$$

and

$$u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial r} - \frac{v^2}{r} + \frac{1}{\rho}\frac{\partial p}{\partial r} = \nu\left(\nabla^2 w - \frac{w}{r^2}\right), \qquad (3)$$

where

$$\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial x^2}. \qquad (4)$$

The continuity equation is

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial r} + \frac{w}{r} = 0. \qquad (5)$$

The boundary-layer type approximation is made in (4) of neglecting $\partial^2/\partial x^2$ in comparison with $\partial^2/\partial r^2 + \partial/r\partial r$. Equations (1), (2) and (3) then reduce from elliptic to parabolic and take the form

$$u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial r} + \frac{1}{\rho}\frac{\partial p}{\partial x} = \nu\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right), \qquad (6)$$

$$u\frac{\partial v}{\partial x} + w\frac{\partial v}{\partial r} + \frac{vw}{r} = \nu\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2}\right) \qquad (7)$$

and

$$u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial r} - \frac{v^2}{r} + \frac{1}{\rho}\frac{\partial p}{\partial r} = \nu\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} - \frac{w}{r^2}\right). \qquad (8)$$

With this approximation the solution of the equations no longer depends on downstream data, only on upstream data and conditions on the axis and boundary. In fact the initial and boundary conditions are:

(1) u, v, w and p given on an upstream section $x = 0$,

(2) $u = v = w = 0$ and p given on the boundary $r = r_0$,

and

(3) $\partial u/\partial r = v = w = \partial p/\partial r = 0$ on the axis $r = 0$.

We require a stable method of numerical solution for equations (5) to (8). The more general problem with conditions given on an outer boundary $r = R(x)$ (solid wall, stream-line trace or arbitrary surface) should then be soluble by the same method, since a change of

* *Mathematics Department, Royal Aircraft Establishment, Farnborough, Hants.*

independent variable from r to $r/R(x)$ reduces this problem to essentially the constant radius problem.

Possible methods of solution proceed step-by-step with the integration in the downstream direction. Four methods are tried: two explicit and two implicit.

Simple explicit method

We can derive from (5) to (8) the following equations giving the x-derivatives in terms of the variables and their r-derivatives:

$$\frac{\partial u}{\partial x} = -\frac{\partial w}{\partial r} - \frac{w}{r}, \tag{9}$$

$$\frac{\partial v}{\partial x} = \frac{\nu}{u}\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2}\right) - \frac{w}{u}\frac{\partial v}{\partial r} - \frac{vw}{ru}, \tag{10}$$

$$\frac{\partial w}{\partial x} = \frac{\nu}{u}\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} - \frac{w}{r^2}\right) - \frac{w}{u}\frac{\partial w}{\partial r} - \frac{1}{\rho u}\frac{\partial p}{\partial r} + \frac{v^2}{ru} \tag{11}$$

and

$$\frac{1}{\rho}\frac{\partial p}{\partial x} = \nu\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) + u\frac{\partial w}{\partial r} - w\frac{\partial u}{\partial r} + \frac{uw}{r}. \tag{12}$$

We now replace the r-derivatives by finite-difference approximations referred to lines r = constant at a spacing of Δr between $r = 0$ and $r = r_0$:

$$\frac{\partial^2 u}{\partial r^2} = \frac{\delta_r^2 u}{(\Delta r)^2} \quad \text{and} \quad \frac{\partial u}{\partial r} = \frac{\mu_r \delta_r u}{\Delta r}, \tag{13}$$

where

$$\delta_r^2 u = u(r + \Delta r) - 2u(r) + u(r - \Delta r)$$

$$2\mu_r \delta_r u = u(r + \Delta r) - u(r - \Delta r);$$

with similar expressions for the other dependent variables. Equations (9) to (12) then become ordinary differential equations in the independent variable x, which can be solved by the Runge–Kutta process proceeding in steps Δx in the downstream direction from the initial line $x = 0$, on which all quantities are known. The boundary conditions on $r = r_0$ and $r = 0$ are easily incorporated during the calculation.

This method is stable for the simple equation

$$\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial r^2} \tag{14}$$

if $\Delta x/(\Delta r)^2 < 0{\cdot}7$ (see, for example, *Modern Computing Methods*, 1961), and, since the second-order equations of the present system are basically of the form

$$u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial r^2}, \quad u\frac{\partial v}{\partial x} = \nu\frac{\partial^2 v}{\partial r^2}, \quad u\frac{\partial w}{\partial x} = \nu\frac{\partial^2 w}{\partial r^2}, \tag{15}$$

we might conjecture that the proposed method would be locally stable if

$$\frac{\Delta x}{(\Delta r)^2} < \frac{0{\cdot}7u}{\nu}. \tag{16}$$

Thus for a uniform mesh we require

$$\frac{\Delta x}{(\Delta r)^2} < \text{minimum value of } \left(\frac{0{\cdot}7u}{\nu}\right); \tag{17}$$

but $u \to 0$ at the boundary and so this inequality cannot be satisfied by a non-zero Δx, however small. We might argue that as we do not integrate actually along the boundary, but only along a line Δr from it, that in the right-hand side of (17) the smallest internal value of u should be used. This permits a reasonably large value of Δx for the example considered below. However, it is found in practice that this method does not remain stable long enough, however small Δx is, to be of practical use.

Improved explicit method

By a change of independent variable we can obtain a more generous inequality corresponding to (17), the right-hand side of which is non-zero even on the boundaries.

We introduce a stream function ψ such that

$$u = \frac{1}{r}\frac{\partial \psi}{\partial r}, \quad w = -\frac{1}{r}\frac{\partial \psi}{\partial x} \tag{18}$$

which identically satisfy the continuity equation (5). A von Mises transformation (Goldstein, 1938 or Schlichting, 1955, for example) is now applied changing the independent variables from x, r to x, ψ. Equations corresponding to (9) to (12) are:

$$\frac{\partial u}{\partial x} = rw\frac{\partial u}{\partial \psi} - ru\frac{\partial w}{\partial \psi} - \frac{w}{r}, \tag{19}$$

$$\frac{\partial v}{\partial x} = \nu\left(r^2 u\frac{\partial^2 v}{\partial \psi^2} + r^2\frac{\partial u}{\partial \psi}\frac{\partial v}{\partial \psi} + 2\frac{\partial v}{\partial \psi} - \frac{v}{r^2 u}\right) - \frac{vw}{ru}, \tag{20}$$

$$\frac{\partial w}{\partial x} = \nu\left(r^2 u\frac{\partial^2 w}{\partial \psi^2} + r^2\frac{\partial u}{\partial \psi}\frac{\partial w}{\partial \psi} + 2\frac{\partial w}{\partial \psi} - \frac{w}{r^2 u}\right) + \frac{v^2}{ru} - \frac{r}{\rho}\frac{\partial p}{\partial \psi} \tag{21}$$

and

$$\frac{1}{\rho}\frac{\partial p}{\partial x} = \nu u\left(r^2 u\frac{\partial^2 u}{\partial \psi^2} + r^2\left(\frac{\partial u}{\partial \psi}\right)^2 + 2\frac{\partial u}{\partial \psi}\right) - ruw\frac{\partial u}{\partial \psi} + ru^2\frac{\partial w}{\partial \psi} + \frac{rw}{\rho}\frac{\partial p}{\partial \psi} + \frac{uw}{r}. \tag{22}$$

We now have an additional equation to determine $r(x, \psi)$:

$$\frac{\partial r}{\partial x} = \frac{w}{u}. \tag{23}$$

These equations can be solved by the same method as above: the axis is taken as $\psi = 0$ and the boundary as $\psi = \psi_0$, and introducing lines ψ = constant at a spacing of $\Delta\psi$ we replace the derivatives with respect to ψ by finite-difference approximations. We again have a system of ordinary differential equations to solve by the Runge–Kutta method.

The second-order equations in these variables corresponding to (6), (7) and (8) are each essentially of the form

$$\frac{\partial u}{\partial x} = \nu r^2 u \frac{\partial^2 u}{\partial \psi^2}, \tag{24}$$

and we conjecture that the method is locally stable if

$$\frac{\Delta x}{(\Delta\psi)^2} < \frac{0\cdot 7}{\nu r^2 u}. \tag{25}$$

Thus, for a uniform mesh, we require

$$\frac{\Delta x}{(\Delta\psi)^2} < \frac{0\cdot 7}{\text{maximum value of } (\nu r^2 u)}. \tag{26}$$

The right-hand side is now always non-zero, and in general permits a reasonably large value of Δx. However, it is found that this method of solution reduces rather than eliminates the instability, even for Δx satisfying (26) by a large margin. Though no rigorous stability analysis is possible for this system of non-linear equations, a very much simplified analysis given below suggests a possible reason for the instability that arises even when (26) is satisfied.

In the calculations of this Section we have in practice to use the dependent variable u^2 in place of u, in order to remove the singularity $\partial u/\partial\psi(=\partial u/ru\partial r) = -\infty$ at $r = r_0$ where $u = 0$. This involves only small changes in (19) to (23) and does not affect the conjectured stability requirement (26).

Applications of explicit methods to a test case

For Hagen–Poiseuille flow in a cylinder $r = r_0$ we have $v = w = 0$; equations (5) to (8) then give the well-known solution for this flow

$$u = \frac{1}{4\rho\nu}\left(-\frac{dp}{dx}\right)(r_0^2 - r^2). \tag{27}$$

A suitable test case for the full equations is provided by the following problem. At $x = 0$ we take the initial flow to be the Hagen–Poiseuille flow above, with the boundary conditions $u = v = w = 0$ and given constant pressure gradient (the same value as in the initial condition (27)) on $r = r_0$ and the usual conditions on the axis $r = 0$. Then, with a stable method of calculation, the velocity distribution should remain unchanged in the downstream direction, with u continuing to be given by (27) and v and w remaining effectively zero. Through rounding and truncation errors small values of v, w, $\partial p/\partial r$ and $\partial u/\partial x$ are inevitably introduced, but with a stable method these errors will remain small and bounded, while with an unstable method they will grow and ultimately swamp the true solution. In practice it is found that w, the radial component of velocity, is particularly prone to early unstable behaviour; in one example, with a large value for Δx, w increased from zero to about 10^{28} in only 20 steps.

The following consistent values are used in the calculations

$$\nu = 0\cdot 2, \quad \rho = 0\cdot 0012,$$

$$\frac{dp}{dx} = -0\cdot 032 \quad \text{along the boundary } r = 1$$

and $$u = \frac{100}{3}(1 - r^2) \quad \text{at} \quad x = 0. \tag{28}$$

The rate of volume flow $= \int_0^1 2\pi r u dr = \frac{50}{3}\pi$ so that $\psi_0 = 25/3$.

We now compare the results given by the two explicit methods. The simple explicit method and the improved version were tried with $\Delta r = 0\cdot 1$ and

$$\Delta\psi = \frac{25}{3} \times \frac{1}{10} = 0\cdot 8333$$

respectively, that is 9 internal pivotal points in the radial direction in each case, and $\Delta x = 0\cdot 04, 0\cdot 2, 1, 5$. Equation (17) gives respectively

$$\Delta x < 0 \quad \text{or} \quad 0\cdot 21,$$

depending on whether the smallest value of u is taken as the zero value on the boundary or the smallest internal value, and equation (26) gives

$$\Delta x < 0\cdot 29.$$

The behaviour of the maximum value of $|w|$ for $\Delta x = 0\cdot 04$ is shown in Table 1. This shows that the (x, ψ) method is only slightly less unstable than the (x, r) method. At the larger values of Δx both are very bad; for example, with $\Delta x = 0\cdot 2$, $|w|$ is 8×10^{-4} with (x, ψ) and $1\cdot 3 \times 10^{-3}$ with (x, r) after only 5 steps. In a longer test the (x, ψ) method gave a maximum $|w|$ of about 10 after 150 steps of the very small step $\Delta x = 0\cdot 015$. These explicit methods are thus clearly unsuitable.

Table 1

Growth of maximum value of $|w|$ for explicit methods with $\Delta x = 0\cdot 04$

	AFTER $5\Delta x$	$10\Delta x$	$15\Delta x$	$20\Delta x$
(x, r) method	$1\cdot 3 \times 10^{-6}$	$7\cdot 4 \times 10^{-6}$	$4\cdot 4 \times 10^{-5}$	$3\cdot 4 \times 10^{-4}$
(x, ψ) method	$1\cdot 8 \times 10^{-6}$	$3\cdot 4 \times 10^{-6}$	$8\cdot 0 \times 10^{-6}$	$7\cdot 2 \times 10^{-5}$

It is apparent for the (x, ψ) method that satisfying the conjectured stability requirement (26) is not sufficient for stability; neither is (17) for the (x, r) method if the minimum value of u is interpreted as the minimum internal value. A possible reason for the failure of the (x, ψ) method is given below.

Implicit methods

It is well known that, for the simple equation (14), an implicit method based on the use of

$$\frac{u(x+\Delta x) - u(x)}{\Delta x} = \frac{a\delta_r^2 u(x+\Delta x) + (1-a)\delta_r^2 u(x)}{(\Delta r)^2}, \tag{29}$$

with $\frac{1}{2} \leqslant a \leqslant 1$, is stable for all values of $\Delta x/(\Delta r)^2$ (Richtmyer, 1957, for example); $u(x)$ is the column vector of the solution at the pivotal points in the r-direction.

Implicit methods can be applied to the system of equations (5) to (8) by introducing finite differences in the x and r directions, with mesh lengths Δx and Δr respectively. For the Crank–Nicolson method, corresponding to $a = \frac{1}{2}$, we use the approximations

$$\frac{\partial u}{\partial x} = \frac{u(x+\Delta x) - u(x)}{\Delta x}, \tag{30}$$

$$\frac{\partial^2 u}{\partial r^2} = \frac{\delta_r^2 u(x+\Delta x) + \delta_r^2 u(x)}{2(\Delta r)^2} \tag{31}$$

and

$$\frac{\partial u}{\partial r} = \frac{\mu_r\delta_r u(x+\Delta x) + \mu_r\delta_r u(x)}{2\Delta r}, \tag{32}$$

with similar expressions for the derivatives of v, w and p.

Another implicit method is the fully implicit method, which, for the simple equation (14), corresponds to taking $a = 1$ in (29). For its application to (5) to (8) we use the approximations

$$\frac{\partial^2 u}{\partial r^2} = \frac{\delta_r^2 u(x+\Delta x)}{(\Delta r)^2} \quad \text{and} \quad \frac{\partial u}{\partial r} = \frac{\mu_r\delta_r u(x+\Delta x)}{\Delta r}, \tag{33}$$

with similar expressions for the r-derivatives of v, w and p. The x-derivatives are still given by equations of the form (30).

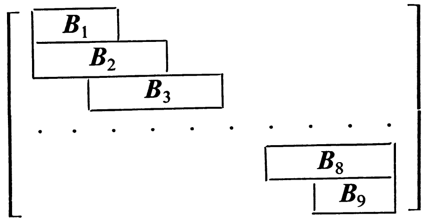
When the substitutions corresponding to either implicit method are made in (5) to (8) we have a system of algebraic equations to solve for each step in the x-direction; each internal pivotal point in the r-direction contributes 4 equations. This system of equations is linearized by taking the coefficients u and w of the non-linear convective acceleration terms, in the finite-difference equivalents of (6) to (8), at their known values for the line x. The terms vw and v^2 are approximated by $\frac{1}{2}[v(x)w(x+\Delta x) + v(x+\Delta x)w(x)]$ and $v(x)v(x+\Delta x)$ respectively.

The boundary conditions on $r = r_0$ ($u = v = w = 0$, p given) are easily incorporated into the difference equations for the line $r = r_0 - \Delta r$, and similarly with $v = w = 0$ on $r = 0$ for the difference equations along $r = \Delta r$. For the conditions $\partial u/\partial r = \partial p/\partial r = 0$ on the axis of symmetry $r = 0$ we assume that parabolae with axes $r = 0$ pass through the values of u and p at $r = \Delta r$ and $2\Delta r$ and hence through the values at the "fictitious" points $r = -\Delta r$ and $-2\Delta r$. This gives the boundary values of u and p in terms of internal values, and the former can then be eliminated.

The system of differential equations has now been replaced by the linear algebraic equations

$$\boldsymbol{A}(x)y(x+\Delta x) = \boldsymbol{b}(x), \tag{34}$$

where $\boldsymbol{A}$ is a square band matrix of bandwidth twelve. The first 4 rows and the last 4 rows of $\boldsymbol{A}$ and $\boldsymbol{b}$ are modified by the boundary conditions, while the remainder of $\boldsymbol{A}$ is diagonal in terms of 4×12 sub-matrices. In all the examples discussed here we take $r_0 = 1$ with $\Delta r = 0{\cdot}1$; there are thus 9 internal pivotal points in the r-direction and $\boldsymbol{A}$ is 36×36 and has the form

$\boldsymbol{A} =$  ,

where the sub-matrix $\boldsymbol{B}_1$ is 4×8, $\boldsymbol{B}_2$ to $\boldsymbol{B}_8$ are 4×12 and $\boldsymbol{B}_9$ is 4×8. The actual elements of the sub-matrices are left unspecified for brevity, but are easily obtained for each implicit method from the relevant difference equations. $y(x + \Delta x)$ in (34) is the column vector of the solution at $x + \Delta x$ and is given in an obvious notation by

$$y = \{u_1, v_1, w_1, p_1;\ \ u_2, v_2, w_2, p_2;\ \ldots;\ \ u_9, v_9, w_9, p_9\}.$$

The column vector $\boldsymbol{b}$ consists of those terms of the difference equations which do not involve the unknowns. The system of linear equations (34) is solved by the direct method of Gaussian elimination.

The accuracy of the calculations can be increased for either implicit method by an iterative procedure: after obtaining the solution of the linearized equations above we replace the coefficients u and w by the means of the values at x and those just found at $x + \Delta x$, and then repeat the solution. It is found in practice that one such iteration, further to the initial calculation, is sufficient.

Near $x = 0$ we may require for accuracy a smaller Δx than is needed for large x, where the flow is perhaps asymptotically approaching a uniform state. With the fully implicit method there is no sign of instability even with Δx as large as $\Delta x = 40$. This is a great advantage of the implicit methods, as the calculations would become very laborious if the asymptotic behaviour was required and a small Δx was necessary for stability.

Approximate stability analysis

Due to the non-linearity of the differential equations it is not possible to give a rigorous stability analysis. However, we give a much simplified one which suggests a possible advantage of the fully implicit over the Crank–Nicolson method in the present application.

We expect instability to arise, if at all, from the momentum equations which are second-order and parabolic, rather than from the first-order continuity equation. The momentum equations are each basically of the form

$$u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial r^2}, \qquad (35)$$

if only the highest derivative is retained for each independent variable. Equation (35) can be linearized to

$$u_*\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial r^2}, \qquad (36)$$

where u_* is at present regarded as constant but is later taken as the local value of u.

The difference formula

$$u_*\left[\frac{u(x+\Delta x)-u(x)}{\Delta x}\right] = \frac{\nu}{(\Delta r)^2}[a\delta_r^2 u(x+\Delta x) + (1-a)\delta_r^2 u(x)] \qquad (37)$$

contains both the Crank–Nicolson method ($a = \frac{1}{2}$) and the fully implicit method ($a = 1$). We have from (37)

$$(\boldsymbol{I} - \beta a\boldsymbol{C})u(x+\Delta x) = [\boldsymbol{I} + \beta(1-a)\boldsymbol{C}]u(x), \qquad (38)$$

where

$$\beta = \frac{\nu\Delta x}{u_*(\Delta r)^2}, \qquad (39)$$

$$\boldsymbol{C} = \begin{pmatrix} -2 & 1 & & & & \\ 1 & -2 & 1 & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & 1 & -2 & 1 \\ & & & & 1 & -2 \end{pmatrix} \qquad (40)$$

and $\boldsymbol{I}$ is the unit matrix. $\boldsymbol{C}$ and $\boldsymbol{I}$ are both $n \times n$ matrices if there are n internal pivotal points in the r-direction. The latent roots of $\boldsymbol{C}$ are

$$\omega_i = -4\sin^2\theta_i \quad \text{where} \quad \theta_i = \frac{i\pi}{2(n+1)}$$

with $i = 1, 2, \ldots n$;

and therefore the latent roots of

$$(\boldsymbol{I} - \beta a\boldsymbol{C})^{-1}[\boldsymbol{I} + \beta(1-a)\boldsymbol{C}]$$

are

$$\lambda_i = \frac{1 + \beta(1-a)\omega_i}{1 - \beta a\omega_i}. \qquad (41)$$

At the boundary $u_* \to 0$ so that $\beta \to \infty$ and

$$\lambda_i \to \frac{1-a}{-a} = 1 - \frac{1}{a}. \qquad (42)$$

Thus, for the Crank–Nicolson method: $|\lambda_i| \to 1$. For the fully implicit method: $|\lambda_i| \to 0$. The former is on the margin of stability and, with the full unsimplified non-linear equations, it is possible that the method could cease to be stable; while for the latter there is little risk of this. A very similar situation is pointed out in Chapter 29 of Fox (1962).

For the explicit method which uses the independent variables (x, ψ) in place of (x, r) we have $a = 0$ and hence

$$\lambda_i = 1 + \beta_1\omega_i, \qquad (43)$$

where β_1, which now replaces β, is given by

$$\beta_1 = \frac{\nu r_*^2 u_* \Delta x}{(\Delta\psi)^2}, \qquad (44)$$

so that $\beta_1 \to 0$ at the boundary and $|\lambda_i| \to 1$. This is again on the limit of stability, and, as for the Crank–Nicolson method, this may be exceeded by the full non-linearized equations. This is suggested as a reason for the instability of this method in practice.

Table 2

Behaviour of maximum values of $|w|$ and $|v|$ for implicit methods

Δx	METHOD	MAXIMUM OF $\|w\|$					MAX. $\|v\|$ AFTER $5\Delta x$
		AFTER Δx	$2\Delta x$	$3\Delta x$	$4\Delta x$	$5\Delta x$	
2	CN	4×10^{-6}	3×10^{-5}	3×10^{-4}	3×10^{-3}	2×10^{-2}	3×10^{-7}
2	FI	3×10^{-6}	7×10^{-6}	2×10^{-5}	5×10^{-5}	2×10^{-4}	6×10^{-7}
5	CN	4×10^{-7}	4×10^{-6}	2×10^{-5}	5×10^{-5}	1×10^{-4}	3×10^{-7}
5	FI	1×10^{-6}	4×10^{-6}	6×10^{-6}	6×10^{-6}	4×10^{-6}	4×10^{-7}
10	CN	4×10^{-7}	2×10^{-6}	6×10^{-6}	1×10^{-5}	3×10^{-5}	4×10^{-7}
10	FI	8×10^{-7}	1×10^{-6}	1×10^{-6}	1×10^{-6}	8×10^{-7}	2×10^{-7}
20	CN	2×10^{-7}	3×10^{-7}	3×10^{-7}	6×10^{-7}	9×10^{-7}	6×10^{-7}
20	FI	3×10^{-7}	1×10^{-7}	6×10^{-8}	1×10^{-7}	1×10^{-7}	4×10^{-8}

Application of implicit methods to test case

Results obtained by the two implicit methods, for the test case of Hagen–Poiseuille flow, are summarized in Table 2. Calculations were done with $\Delta x = 2, 5, 10, 20$, and the table gives the value of the maximum of $|w|$ occurring at the end of each of the first 5 steps and the maximum of $|v|$ after the 5th step.

The value of p was correct to 4 significant figures in all cases except the first, which had a maximum error of 1 in the 4th place at the 5th step. The values of maximum $|v|$ were always small and decreased with increasing x in all cases. The value of u was correct to 4 significant figures in all cases with the exception again of the first one, which had a maximum error of 4 in the 3rd place after 5 steps. In general w varied rather erratically with r, while v was smooth.

It can be seen from the table that the fully implicit method is satisfactory for $\Delta x = 5, 10, 20$ but the Crank–Nicolson method is only satisfactory for $\Delta x = 20$. It is not known why there should be this difficulty for small Δx, nor why the Crank–Nicolson method is so much worse in this respect than the fully implicit method.* In view of this, and the difference between the limits of the latent roots, we use the fully implicit method for the example of the next Section.

Example

We consider a viscous flow in a cylinder $r = 1$; at the initial section $x = 0$ the flow is now taken as

$$u = 20 + 20r^2 - 40r^4, \quad v = 20r(1 - r) \qquad (45)$$

with $w = 0$ and $\partial p/\partial r = 0$ as before. The boundary conditions along $r = 0$ and $r = 1$ are unchanged, in particular the pressure gradient $(-\partial p/\partial x)$ along $r = 1$ is $0 \cdot 032$, and the rate of volume flow with (45) is the same as that of the Hagen–Poiseuille flow with this pressure gradient.

Figs. 1 and 2 show how the flow tends to the Hagen–

* It is possible that this behaviour is related to the fact that for small Δx β is relatively small and the latent root λ_1, corresponding to θ_1 the smallest value of θ_i, is near 1 for both methods. For example with $\Delta x = 0 \cdot 1$ and $\Delta r = 0 \cdot 1$ we have

$$\beta_{min} = 0 \cdot 06, \quad \theta_1 = 9° \quad \text{so that} \quad \sin^2 \theta_1 = 0 \cdot 02447$$

and $\lambda_1 = 0 \cdot 994$ for both $a = \frac{1}{2}$ and $a = 1$. However, this fails to explain why $a = \frac{1}{2}$ is worse in practice than $a = 1$.

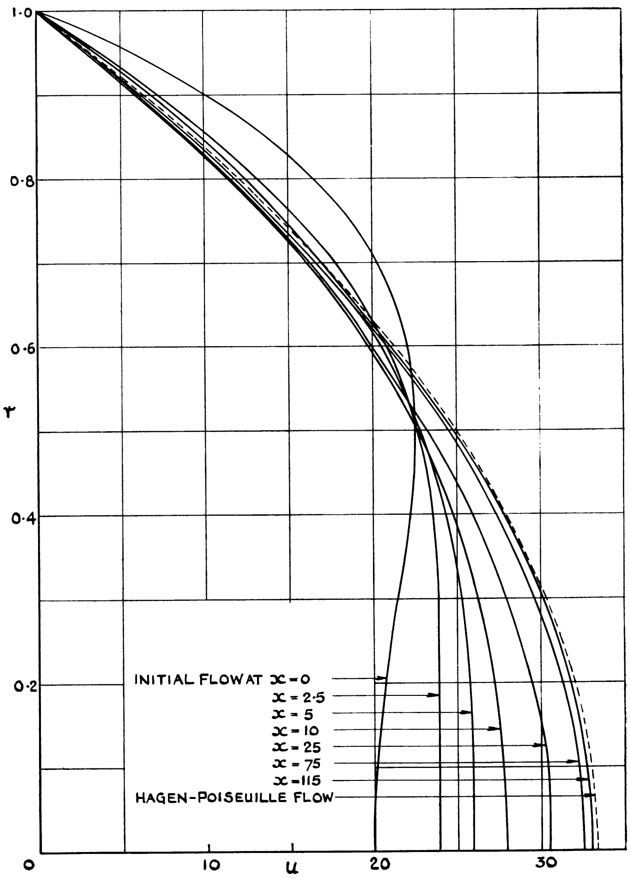


Fig. 1.—Profile of axial velocity

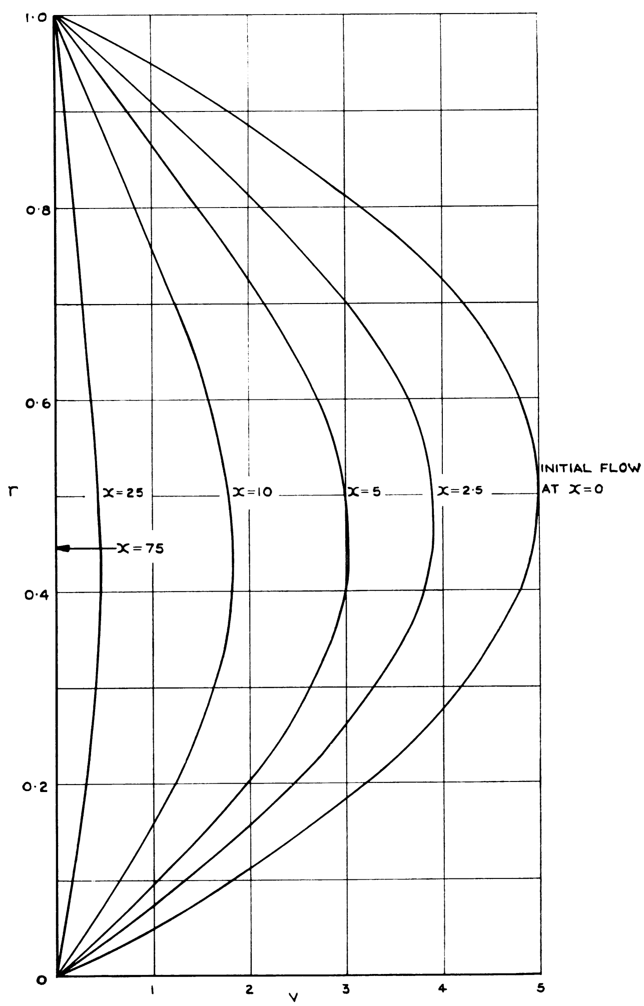


Fig. 2.—Profile of circumferential velocity

Poiseuille flow as x increases: v is reduced to zero and u becomes parabolic.

A further calculation was carried out with the values of (45), but a small non-zero value of w, proportional to $r(1 - r)$, was introduced at $x = 0$. w was eventually damped out and the same uniform state was reached as before.

The program for these problems was written in Mercury Autocode. Each step of the solution in the x-direction took about 90 seconds.

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Book reviews: Mathematical tables

Handbook of Mathematical Tables. Edited by S. M. SELBY, 1962; x + 580 pages. (Cleveland, Ohio: *Chemical Rubber Publishing Co.*,$ 7.50 in U.S.A., $8 outside U.S.A.)

This book of tables originated as a section in the *Handbook of Chemistry and Physics*, which has passed through many editions. It has now, for the first time, become a fully-fledged book, still labelled as *Supplement* to the *Handbook*.

It is impossible to do justice to this large and widespread collection of material in a brief notice; one can only pick out some characteristic points.

There are the usual familiar tables of elementary functions, but 4-, 5- and 6-place tables of Common Logarithms are *all* present, for the best convenience of a variety of users, as well as a table of Natural Logarithms. Trigonometric and Circular Functions are given for arguments in degrees and minutes, in degrees and decimals, and for radian argument, and other familiar functions, Powers, Roots, Reciprocals, etc., are also represented, as also are Factors, Primes and Interest Tables.

There are Statistical Tables including t- and F-tests for Significance, and tables of the Binomial and Poisson Distributions. Then there are tables of the Gamma, and Elliptic functions, of Sine, Cosine, Exponential Integrals, of Legendre, and some Bessel Functions.

Besides the Tables there are extensive lists of formulae in many branches of mathematics—we mention the substantial Table of Integrals, Formulae in Algebra, including Algebra of Sets, Integral Domains, Groups, Fields and Rings, and Series expansions, etc., and many Trigonometrical formulae.

A special section on Planetary Orbits is perhaps topical in the space age.

Perhaps the items listed will give some small idea of the wealth of material contained. The book is substantial, well printed and well bound, and many users will find it useful.

J. C. P. MILLER.

Tables of Lamé Polynomials, by F. M. ARSCOTT and I. M. KHABAZA, 1962; xxxii + 79 + 66 + 66 + 66 + 66 + 66 + 66 + 52 pages. $8\frac{1}{2}$ in. × 11 in. (Oxford, London, New York and Paris: *Pergamon Press*, £7.)

This is a modern set of tables produced on an automatic computer—in fact a Ferranti Mercury—and reproduced photographically from sheets produced by the computer.

The tables are preceded by a brief summary of the Theory of Lamé polynomials, and short notes on the method of computation and use of the tables, and on the computer program.

The equation

$$\frac{d^2w}{dz^2} + \left\{h - n(n+1)k^2\,\mathrm{sn}^2\,z\right\}w = 0$$

possesses solutions in finite terms when n is an integer and h has one of a set of $(2\,n + 1)$ eigenvalues. There are eight types

$$w = \mathrm{sn}^{\rho}\,\mathrm{cn}^{\sigma} z\,\mathrm{dn}^{\tau} z\,F(\mathrm{sn}^2 z)$$

with each of ρ, σ, τ independently 0 or 1 and $F(x)$ a polynomial in x, of degree $\frac{1}{2}(n - \rho - \sigma\ \ \tau)$. Here sn z, cn z, dn z are Jacobian elliptic functions.

The tables give 6-figure values of the coefficients in these polynomials, for degrees n up to 30, with $C = k^2 = 0{\cdot}1\,(0{\cdot}1)\,0{\cdot}9$ for each of the eight types. Also given is the corresponding eigenvalue for each polynomial.

The layout of the tables could be improved. For purposes of programming easily, a standard layout has been chosen so that the automatic computer could produce the final pages directly. It is, however, more wasteful of space than it might have been and so less compact and thus a little less easy to use. The layout is, however, straightforward and quite clear —which is a distinct advance on a number of computer-produced tables.

The photographic reproduction leaves something to be desired; there is some variation in blackness, but the main defect is a variation in thickness and a woolly look to many of the printed digits, possibly due to non-uniformity of the copy from which the tables were produced. The figures, with head-and-tail type, are, however, very legible.

This book usefully places these polynomials on record, and refers to an extension of the tables, for degrees n from 31 to 60, held in the Depository of Unpublished Mathematical Tables at the Royal Society.

All the more comprehensive libraries of mathematical tables should have this book of standard, though fairly advanced, functions on their shelves. J. C. P. MILLER.