A computer technique for game-theoretic problems I: Chemin-de-fer analyzed

By F. G. Foster*

A numerical technique will be described for solving a class of $N \times 2$ matrix games, where N is very large. The method of computing the value of the game of *chemin-de-fer* and the results of a computation carried out on the Atlas computer will be presented.

1. The theory of games

Game theory considers situations in which there are two (or more) competing persons whose actions influence but do not individually (or possibly jointly) completely determine the outcome of a certain event. Depending on the outcome, the players receive certain payments. If in the case of two players for each possible event one player wins exactly what the other loses, the game is called a *two-person zero sum* game, or simply a *matrix game*.

Usually the players will not agree as to which event should occur, i.e. their objectives are different. The question then arises for each player as to what his best actions are in the various situations in the game. In the case of a matrix game, game theory provides a solution, based on the principle that each player tries to choose his course of action so that, *regardless of what his opponent does, he can assure himself of a certain amount.*

Many games of strategy like chess, bridge or poker can be translated into the form of matrix games. Let us take first a very simple example.

Each of two players A and B hold up simultaneously either one or two fingers. A pays B an amount equal to the total number of fingers shown. The game can be depicted in matrix form

		A: I	A: II
B	I	2	3
	II	③	4

The elements show the *pay-offs* to B. The game is zero-sum, since A loses what B wins. How should the players play this game?

If A plays the first column, he loses 2 or 3. If he plays the second column, he loses 3 or 4. Obviously, he will play I. If B plays the first row, he wins 2 or 3, if he plays the second row he wins 3 or 4. Thus he will play II.

We call "play I" or "play II" a *strategy*. Thus I is A's *optimal strategy* and II is B's optimal strategy. By playing I, A assures himself of a loss of at most 3. By playing II, B assures himself of a gain of at least 3. We call 3 the *value* of the game, being the outcome when each player uses his optimal strategy.

Note that 3 is the minimum of the row in which it occurs and the maximum of the column.

Suppose the matrix were as follows:

		A: I	A: II
B	I	2	5
	II	3	4

What are the optimal strategies?

Clearly A must, if he is rational, choose I. Now B would like to gain 5, but if he chooses I, he will surely only gain 2. Therefore B will choose II. Thus the optimal strategies and value of the game are as previously. Note again that 3 remains the minimum of the row and the maximum of the column.

Such a game is called *strictly determined* if the matrix contains an element which is simultaneously a row-minimum and a column-maximum. This is the value of the game. The game is *fair* if its value is zero.

2. Non-strictly determined games

Suppose the game were that A pays B one unit if the number of fingers held up is the same, and B pays A if they differ. Then the pay-off matrix to B would be

		A: I	A: II
B	I	1	−1
	II	−1	1

This matrix does not have any element which is a row-minimum and a column-maximum. How should one

* *London School of Economics and Political Science, London, W.C.2*

play such a *non-strictly determined* game? Clearly no one choice of strategy can be optimal. If A always plays I, B will find this out and play I also. Similarly with II. The answer is the players must randomize, and adopt mixed strategies. By a mixed strategy is meant a rule of the form: play I with probability h and II with probability $(1 - h)$. The value of h has to be calculated on the basis of the given pay-off matrix. In this particular example, obviously by symmetry $h = \frac{1}{2}$.

For the general 2×2 pay-off matrix,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

the value of h will be computed by solving

$$v = ha + (1 - h)b = hc + (1 - h)d$$

for h. The value of v is the value of the game. Geometrically, we have to find the point of intersection of two lines; as in Fig. 1.

The value of such a non-strictly determined game is an *expected value*. v is the largest expectation of gain that B can assure himself of, and likewise it is the smallest expectation of loss that A can assure himself of. The *fundamental theorem* of game theory is that any $m \times n$ matrix game has a unique value so defined. The corresponding optimal (mixed) strategy for either player is called a *minimax* strategy.

3. $N \times 2$ matrix games

If A has 2 strategies and B has N strategies, we get an $N \times 2$ matrix. Its value is obtained by looking in turn at ${}^{N}C_2$ pairs of lines, and obtaining the greatest value in the ${}^{N}C_2$ 2×2 matrix games.

An alternative procedure is to draw the piece-wise linear curve below which all pay-offs for given h lie. Let $G(h, s)$ be the gain to B corresponding to h and to B's pure strategy s. Then the curve as a function of h is given by

$$G(h) = \max_{s} G(h, s).$$

The value of the game is given by the minimum point on the curve. See Fig. 2.

The question arises as to whether this is a feasible numerical method in practice to compute $G(h)$ and its minimum point. In fact, it turns out that it can be used in certain cases where B's strategies are such that they prescribe for each of a number of situations one of a small number of choices. Suppose, for example, there are n situations in each of which there is for B a binary choice. Then $N = 2^n$. Now the gain for B following any particular one of these for given h is very often composed of additive components: in situation i the component is either a_i^0 or a_i^1 depending on the binary choice. Thus the total gain is

$$G(h, s) = a_1^{(c_1)} + a_2^{(c_2)} + \ldots + a_n^{(c_n)}$$

where $(c_1, c_2, \ldots c_n)$ represents the strategy, s,

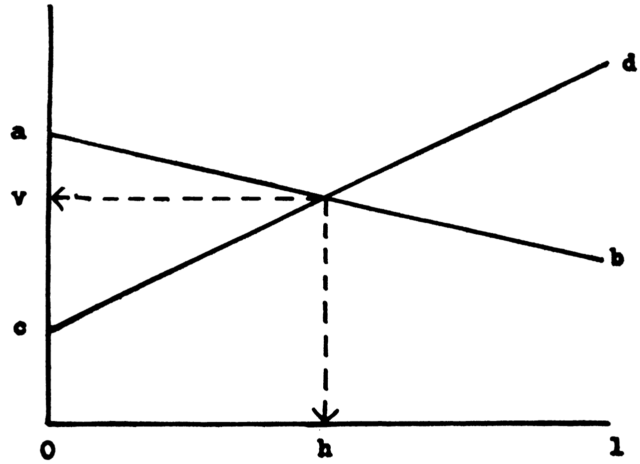

Fig. 1.—Graph showing the value v of a 2×2 matrix game and the frequency h for A's minimax strategy

$(c_j = 0 \text{ or } 1)$. Thus to compute $\max_s G(h, s)$ it is necessary only to make $2n$ calculations by examining in turn a_i^0 and a_i^1 and selecting the greater.

The method will be illustrated by a study of the game of chemin-de-fer.

4. The rules of chemin-de-fer

There are two players, A and B, and B is called the *Banker*. They are each dealt face-down two cards from a deck consisting of 6 packs of playing cards. They add the values of the two cards, and the resulting scores constitute their *initial score*. The way in which the values of the two cards are added is as follows. Addition is modulo 10 and J, Q, K count as 10. Thus:

$$3 + 9 = 2$$
$$J + 7 = 7$$
$$10 + J = 0.$$

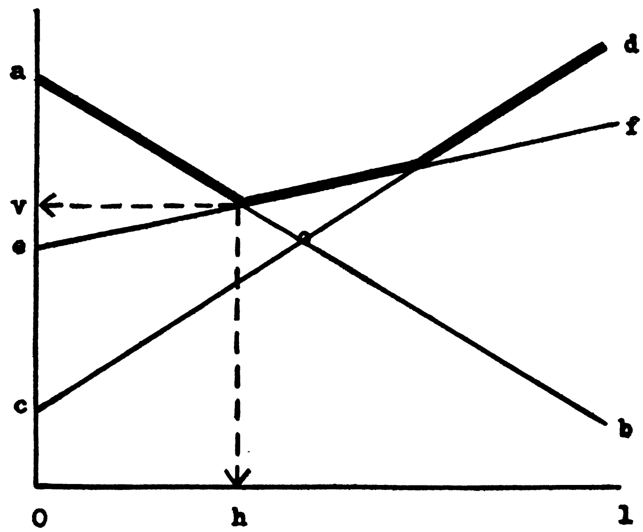

Fig. 2.—Graph showing the value v of a 3×2 matrix game and the frequency h for A's minimax strategy

A makes the first move. If A has an initial score of 8 or 9, he shows his cards, and B then also shows his whatever they are. The player with the higher score wins. If the scores are equal, they draw.

The stake is placed before the cards are dealt. We can assume there is unit stake on either side which the winner collects.

If A has an initial score of 0, 1, 2, 3 or 4, he must call for another card, and he adds its value to his initial score (again mod 10). It is dealt *face up*. If A has an initial score of 6 or 7, he must not call for another card, but stand. If A has a 5, he has to decide whether to call or stand.

The Banker B is in a slightly different situation. He observes A's move, i.e. to stand or call, and if A calls, he observes the value of the card dealt to A. He does not of course know A's initial score, but he does know that if A calls he must have a 0, 1, 2, 3, 4 or 5, and if A stands he must have a 5, 6 or 7. (If A had an 8 or 9 he would have already declared his hand.)

The Banker B, with this additional information, including of course a knowledge of his own initial score, has to respond to A's move. The rules are:

If B has 0, 1 or 2, he must call for another card.

If B has a 7, he must stand.

If B has 3, 4, 5 or 6, he has to decide whether to call or stand.

If he has an 8 or 9 he turns up his hand, and wins the game, unless of course A has already declared his 8 or 9, in which case he wins or loses in accordance with the rule given for A having 8 or 9.

5. The matrix game

Let us now translate this game specified by its rules into a matrix game.

A *pure strategy* for a player is the complete set of instructions which tells him what to do in every conceivable situation that can arise in the game when he has a choice. Thus A has only two pure strategies which are:

(1) Call on a 5.
(2) Stand on a 5.

For B, however, there is a very large number indeed of pure strategies. Thus he must know what to do when his initial point is $j\,(3 \leqslant j \leqslant 6)$ and when he observes A's further move (of calling or standing) and if A calls, the value of the further card drawn. Thus he himself is in one of 4 different states, and he observes A in one of 11 different states (standing, or drawing 0, 1, . . . 9). Therefore, he has a choice in each of 44 different situations, and each is a binary choice: to call or stand. There are, therefore, for B, 2^{44} pure strategies (or approximately 2×10^{13}).

We can represent any one of these in the form of a table. For this purpose denote (purely conventionally) by "10", the event that A stands. Then the moves that B observes A make may be denoted by

$$l = 0, 1, 2, \ldots 9, 10.$$

B himself is in state $j = 3, 4, 5$ or 6. (In other states he has no choice.) Thus a pure strategy for B might be as given in Table 4. Let us consider now how we might compute the value and optimal strategies for this game.

6. Mathematical formulation

Denote by i A's initial score. Then the possible values are $i = 0, 1, 2, \ldots, 9$. Denote by a_i the probability of an initial score of i. Denote by j B's initial score. Then $j = 0, 1, 2, \ldots, 9$ with the same probabilities as for A.

Now the probabilities, a_i, are easily calculated. For example, an initial score $i = 0$ is obtained with the following combinations of points:

FIRST CARD	SECOND CARD	NO. OF FAVOURABLE COMBINATIONS
0	0	96×95
1	9	24×24
2	8	24×24
3	7	24×24
4	6	24×24
5	5	24×23
6	4	24×24
7	3	24×24
8	2	24×24
9	1	24×24

The total number of combinations $= 312 \times 311$.
We have from the above

$$a_0 = \frac{96 \times 95 + 24^2 \times 8 + 24 \times 23}{312 \times 311} = \frac{595}{4043}.$$

The other a_i's can be computed in a similar way. The values are given in Table 1 to 10 decimal places.

Note that in the computation of the a_i's we retain the accuracy of the assumption of sampling without replacement of the first card of the pair. However, since in the game of chemin-de-fer the cards are drawn from a shuffled deck of six packs of 52 cards, there will be negligible loss in accuracy in assuming sampling with replacement in the further analysis of the problem. This assumption will now be made, since otherwise the mathematical formulation would be extremely complex.

Table 1

Initial score probabilities

i	a_i
0	0·1471679445
1, 3, 5, 7 or 9	0·0949789760
2, 4, 6 or 8	0·0944842938

Now let us denote by l, m the values of the further points drawn by A and B respectively. If, as mentioned in the previous Section, we denote conventionally by "10" the event that A (or B) does not draw a third card, then l and m take the range of values 0, 1, 2, . ., 10. Denote by b_0, b_1, . ., b_9 the probability, conditional upon drawing, of l (or m) taking the values 0, 1, 2, . ., 9. Define $b_{10} = 0$.

The probabilities, b_l, are also easily calculated. There are 16 cards in each 52 which give a point of 0. Therefore $b_0 = \frac{16}{52}$. For any other point, $l = 1, 2, .., 9$, there are 4 cards in the 52. Therefore $b_l = \frac{4}{52}$. These values are given to 10 decimal places in Table 2.

We have seen that A has two pure strategies: to draw or not to draw for a further point when his initial score is 5. Let A mix his two strategies so that he draws with frequency h.

Denote by p_{il} the probability that A's further point is l (0, 1, 2, . ., 10), given that his initial score is i. Define

$$\delta_l = \begin{matrix} 0 & \text{for } l = 0, 1, 2, .., 9 \\ 1 & \text{for } l = 10. \end{matrix}$$

Then we have

$$\begin{aligned} p_{il} &= b_l \text{ for } i = 0, 1, .., 4 \\ &= hb_l + (1 - h)\delta_l \text{ for } i = 5 \\ &= \delta_l \text{ for } i = 6, 7, 8, 9. \end{aligned}$$

Now denote by $\pi^{(s)}_{lm}$ the probability that B's further point is m when his initial point is j, when he observes A's further point of l and when he uses a certain pure strategy, s. For such a strategy, s, there are two possibilities for each pair of values j, l:

$$\begin{aligned} &\textit{either} \quad \pi^{(s)}_{jlm} = b_m, \quad m = 0, 1, .., 10 \\ &\textit{or} \quad \pi^{(s)}_{lm} = \delta_m, \quad m = 0, 1, .., 10. \end{aligned}$$

For any individual coup, i.e. play of the game, the actual value of i, j, l, m determines the pay-off to B. Denote this pay-off by $f(i, j, l, m)$. Then f is defined by the following rules:

The amount staked is one unit. Let all addition be modulo 10.

For $j = 8$ or 9

$$\begin{aligned} f &= -1 \quad \text{for } i > j \\ &= 0 \quad \text{for } i = j \\ &= 1 \quad \text{for } i < j. \end{aligned}$$

For $j \neq 8$ or 9

$$\begin{aligned} f &= -1 \quad \text{for } i = 8 \text{ or } 9 \\ &\quad \text{or for } i + l > j + m, i \neq 8 \text{ or } 9 \\ f &= 0 \quad \text{for } i + l = j + m, i \neq 8 \text{ or } 9 \\ &= 1 \quad \text{for } i + l < j + m, i \neq 8 \text{ or } 9. \end{aligned}$$

We can now write down an expression for the expected pay-off to B, $G(h, s)$, expectation being taken over all possible coups (i, j, l, m):

Table 2

Further point probabilities

l	b_l
0	0·3076923076
1, 2, . ., 9	0·0769230769

$$G(h, s) = \sum_{i=0}^{9} \sum_{j=0}^{9} \sum_{l=0}^{10} \sum_{m=0}^{10} a_i p_{il} a_j \pi^{(s)}_{jlm} f(i, j, l, m).$$

Note that this expression is linear in h, as it should be, the only term involving h being p_{5l}.

In theory, then, we should compute the expected pay-offs for all possible strategies, s, putting $h = 0$ or 1, in order to compute the $N \times 2$ pay-off matrix. This would not, however, be practicable for N very large (as it is), and there are two respects in which the amount of computation can be reduced.

(1) The first reduction results from the observation that for any given h the total expected pay-off is additive over the possibilities (j, l). Thus we can examine for each (j, l) the results of B's drawing or not drawing, i.e. the result of substituting for $\pi^{(s)}_{jlm}$ in turn b_m and δ_m. We select the greater of the two values obtained in this way, and note this contribution to the total pay-off. By doing this systematically over all (j, l), we obtain the maximum pay-off and at the same time the pure strategy that corresponds to this maximum pay-off.

The process may be repeated for a range of values of h, to obtain graphically the piece-wise linear curve discussed in §3. Each point plotted on the graph represents the maximum pay-off obtainable by B, and corresponding to it the strategy is recorded. It is a simple matter then to obtain graphically the minimum point on the curve. This will be the value of the game. The corresponding value of h represents A's minimax strategy, and B's minimax strategy can be computed for this interpolated value of h.

(2) The above represents a feasible program. However, the computation can be further reduced very substantially by taking into account all possible symmetries present in the game. Suppose that B had precisely the same strategies as A. Let A choose his mixed strategy with parameter h, and let B ignore the additional information on l, and also choose the same mixed strategy. Then clearly the pay-off is zero, the game being symmetrical. Thus we have

$$\sum_{i=0}^{9} \sum_{j=0}^{9} \sum_{l=0}^{10} \sum_{m=0}^{10} a_i p_{il} a_j p_{jm} f(i, j, l, m) = 0.$$

Define $d_{jlm} = \pi_{jlm} - p_{jm}$, noting that $\sum_{m=0}^{10} d_{jlm} = 0$, since we are summing over two probability distributions.

Moreover,

$$d_{jlm} = 0 \quad \text{for } j < 3$$
$$\text{and for } j > 6.$$

For we have,

$$\pi_{jlm} = b_m = p_{jm} \text{ for } j < 3$$
$$\pi_{jlm} = \delta_m = p_{jm} \text{ for } j > 6.$$

Therefore we can write the pay-off for given h and B's given strategy, s, as

$$G(h, s) = \sum_{i=0}^{9} \sum_{j=3}^{6} \sum_{l=0}^{10} \sum_{m=0}^{10} a_i p_{il} a_j d_{jlm} f(i, j, l, m)$$
$$= \sum_{j=3}^{6} a_j \sum_{l=0}^{10} \sum_{i=0}^{9} a_i p_{il} \sum_{m=0}^{10} d_{jlm} f(i, j, l, m).$$

But for $j \neq 8, 9$ and $i = 8$ or $9, f(i, j, l, m) \equiv -1 =$ constant. Therefore, for $i = 8$ or 9 in the above expression we have,

$$\sum_{m=0}^{10} d_{jlm} f(i, j, l, m) = 0.$$

Therefore

$$G(h, s) = \sum_{j=3}^{6} a_j \sum_{l=0}^{10} \sum_{i=0}^{7} a_i p_{il} \sum_{m=0}^{10} d_{jlm} f(i, j, l, m).$$

Write

$$g_{jl} = \sum_{i=0}^{7} a_i p_{il} \sum_{m=0}^{10} d_{jlm} f(i, j, l, m).$$

Then

$$G(h, s) = \sum_{j=3}^{6} a_j \sum_{l=0}^{10} g_{jl},$$

and we analyse separately these components, g_{jl}, to the total pay-off, $G(h, s)$. We now look for the maximum pay-off, $G(h)$, over all strategies, s: $G(h) = \max_s G(h, s)$.

For each component, g_{jl}, in $G(h)$ there are two alternatives, according to whether B draws or stands on the pair (j, l). Denote these by $g_{jl}^{(1)}, g_{jl}^{(0)}$. Now defining,

$$d_{jlm}^{(1)} = b_m - p_{jm},$$
$$d_{jlm}^{(0)} = \delta_m - p_{jm},$$

we have for $p = 1$ or 0,

$$g_{jl}^{(p)} = \sum_{i=0}^{7} a_i p_{il} \sum_{m=0}^{10} d_{jlm}^{(p)} f(i, j, l, m).$$

Define $\quad \bar{g}_{jl} = \max(g_{jl}^{(1)}, g_{jl}^{(0)}).$

Then $\quad G(h) = \sum_{j=3}^{6} a_j \sum_{l=0}^{10} \bar{g}_{jl}.$

The computational method is now to take in turn all pairs (j, l) for $j = 3(1)6$ and $l = 0(1)10$, compute $\bar{g}_{jl}$ and enter $p = 1$ or 0 accordingly in the 4×11 decision matrix for B's strategy. The values, $a_j \bar{g}_{jl}$, are accumulated over l and then over j to obtain finally $G(h)$. The decision matrix and this value are printed out, together with the value of h taken. The computation is repeated over a range of values of h.

In this computation let us consider in more detail the separate cases for $j = 3, 4, 5, 6$, with a fixed l.

First define

$$z_{jl} = \sum_{i=0}^{7} a_i p_{il} \sum_{m=0}^{10} (\delta_m - b_m) f(i, j, l, m).$$

Now for $j = 3$ or 4 we have,

$$d_{jlm}^{(1)} = b_m - b_m = 0,$$
$$d_{jlm}^{(0)} = \delta_m - b_m.$$

It follows that $g_{jl}^{(1)} = 0$ and $g_{jl}^{(0)} = z_{jl}$, which could be either positive or negative. Thus

$$\bar{g}_{jl} = \max(z_{jl}, 0),$$

and we do not have to compute both $g_{jl}^{(1)}$ and $g_{jl}^{(0)}$. If $z_{jl} > 0$, we record this as the contribution to $G(h)$, and print $p = 0$ in the decision matrix. Otherwise, we record a zero contribution to $G(h)$, and print $p = 1$.

For $j = 6$, we have

$$d_{jlm}^{(1)} = b_m - \delta_m,$$
$$d_{jlm}^{(0)} = \delta_m - \delta_m = 0,$$

so that $g_{jl}^{(0)} = 0$ and $g_{jl}^{(1)} = -z_{jl}$, which again could be positive or negative. Thus

$$\bar{g}_{jl} = \max(-z_{jl}, 0).$$

Again, we need only compute z_{jl}. If $z_{jl} > 0$, we record a zero contribution to $G(h)$ and print $p = 0$. Otherwise, we record $-z_{jl}$ as the contribution and print $p = 1$ in the decision matrix.

The case $j = 5$ is slightly different. We have

$$d_{jlm}^{(1)} = b_m - hb_m - (1-h)\delta_m$$
$$= (1-h)(b_m - \delta_m),$$
$$d_{jlm}^{(0)} = \delta_m - hb_m - (1-h)\delta_m$$
$$= h(\delta_m - b_m).$$

Therefore $\quad g_{jl}^{(1)} = (1-h)(-z_{jl})$

$$g_{jl}^{(0)} = hz_{jl}.$$

However, again we need only compute z_{jl}. If $z_{jl} > 0$, we record hz_{jl} as the contribution to $G(h)$ and print $p = 0$ in the decision matrix. Otherwise, we record $(h-1)z_{jl}$ and print $p = 1$.

7. Numerical results

The computation described was programmed for the Ferranti Atlas computer. It was, in fact, only part of a larger computation to obtain the Banker's gain in the game of baccarat. However, the chemin-de-fer calculation provides a neater illustration of the computer technique. The baccarat calculation is a straightforward generalization, but more complex, and the results of it will be described in a subsequent paper. The numerical results for chemin-de-fer alone will be presented here. Since the computation described was an integral part of

Table 3

Banker's expected gain, $G(h)$, by adoption of an optimal strategy against given h

h	0	0·25	0·5	0·75	0·775	0·8	0·85	0·9	1·0
$G(h)$	1·539	1·453	1·373	1·302	1·294	1·287	1·298	1·322	1·371

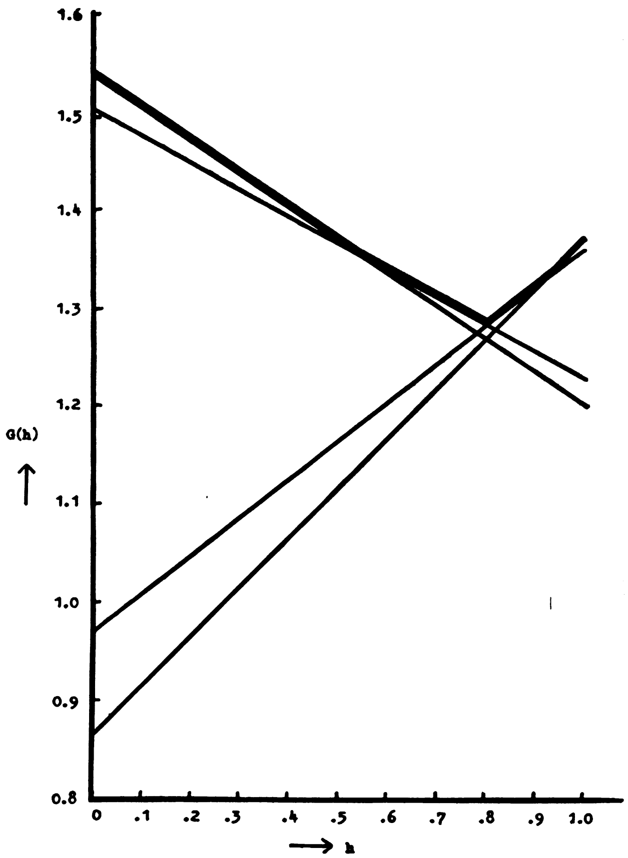

Fig. 3.—Piece-wise linear graph of Banker's expected gain $G(h)$ in chemin-de-fer, by adoption of an optimal strategy against given h

a larger computation, separate timings were not made. It is, however, roughly of the order of 5 seconds per value of $G(h)$.

In all, nine values of h were taken in the range 0 to 1, and the results plotted. The values were obtained sequentially until the piece-wise linear curve could be accurately drawn. No attempt was made to construct a sophisticated program for carrying out this procedure, although it is not difficult to see how the piece-wise linear nature of the curve could make this possible. The values of $G(h)$ computed are given in Table 3 and the corresponding graph is drawn in Fig. 3. The values of $G(h)$ are percentages, i.e. they represent the Banker's expected gain on a stake of 100 money units. It will be seen that the minimum point occurs at $h = 0{\cdot}8$ approximately, for which the gain is approximately 1·29%.

Table 4

Decision table giving Banker's optimal strategy for h in the range $0 \leqslant h \leqslant 0{\cdot}5$

	0	1	2	3	4	5	6	7	8	9	10
3	1	1	1	1	1	1	1	1	0	0	1
4	0	0	1	1	1	1	1	1	0	0	1
5	0	0	0	0	1	1	1	1	0	0	1
6	0	0	0	0	0	0	1	1	0	0	0

Table 5

Decision table giving Banker's optimal strategy for h in the range $0{\cdot}5 \leqslant h \leqslant 0{\cdot}8$. The ringed number indicates a change from the previous table

	0	1	2	3	4	5	6	7	8	9	10
3	1	1	1	1	1	1	1	1	0	(1)	1
4	0	0	1	1	1	1	1	1	0	0	1
5	0	0	0	0	1	1	1	1	0	0	1
6	0	0	0	0	0	0	1	1	0	0	0

Table 6

Decision table giving Banker's optimal strategy for h in the range $0{\cdot}8 \leqslant h \leqslant 0{\cdot}9$. The ringed number indicates a change from the previous table

	0	1	2	3	4	5	6	7	8	9	10
3	1	1	1	1	1	1	1	1	0	1	1
4	0	0	1	1	1	1	1	1	0	0	1
5	0	0	0	0	1	1	1	1	0	0	1
6	0	0	0	0	0	0	1	1	0	0	(1)

Between $h = 0$ and $h = 0{\cdot}5$, the graph is a straight line. This means that the Banker's optimal strategy does not alter for this range of h. This strategy is given in Table 4. Between $h = 0{\cdot}5$ and $h = 0{\cdot}8$, the optimal strategy changes to that shown in Table 5. It will be noted that the only difference is in cell (3, 9), which has changed from 0 to 1. Between $h = 0{\cdot}8$ and $h = 0{\cdot}9$, the optimal strategy is changed to that shown in Table 6 with the difference that cell (6, 10) has now changed from 0 to 1.

Table 7

Decision table giving Banker's optimal strategy for h in the range $0 \cdot 9 \leqslant h \leqslant 1$. The ringed number indicates a change from the previous table

	0	1	2	3	4	5	6	7	8	9	10
3	1	1	1	1	1	1	1	1	0	1	1
4	0	(1)	1	1	1	1	1	1	0	0	1
5	0	0	0	0	1	1	1	1	0	0	1
6	0	0	0	0	0	0	1	1	0	0	1

Between $h = 0 \cdot 9$ and $h = 1 \cdot 0$, the optimal strategy is changed to that shown in Table 7, in which cell (4, 1) is now 1. In all, there are thus only four pure strategies involved for B.

The Banker's minimax strategy is now seen to be a mixed strategy obtained from those of Table 5 and Table 6. The frequency with which the former should be chosen is easily computed from the graph to be approximately $0 \cdot 6$. In other words, the strategy should be as given in Table 5, except that cell (6, 10) should be randomized: with frequency $0 \cdot 6$ the Banker should stand on (6, 10).

The corresponding optimal strategy* for the player we have seen is to call on a 5 with frequency $0 \cdot 8$.

8. Previous computations of chemin-de-fer

The Banker's gain in chemin-de-fer has been of interest to gamblers for many years, and several manual attempts have been made for particular values of h. Scarne (1961), for example, describes a calculation he did taking $h = 0 \cdot 5$, "covering many sheets of paper and taking many days to complete." The result agrees to two significant figures with that obtained here. A manual calculation for $h = 0$ and $h = 1$ will also be described in a forthcoming paper by Kendall and Murchland.

* It is an interesting fact that this strategy is often attained approximately in practice by the device of standing on the pair 2, 3 and calling on any other combination adding to 5: this gives approximately the right frequency of calling. However, the reason why it is optimal is certainly not known. It has been pointed out to me that the same phenomenon has been observed by computer manufacturers in the field of process control, where operatives frequently achieve manually nearly optimal working without apparent calculation.

References

SCARNE, J. (1961). *Scarne's Complete Guide to Gambling*, Simon and Schuster.

KENDALL, M. G., and MURCHLAND, J. D. "Statistical Aspects of the Legality of Gambling," *J. Royal Stat. Soc., Series A*. To be published.

Book review: Threshold decoding

Threshold Decoding, by JAMES L. MASSEY, 1963; 129 pages. (Cambridge, Massachusetts: *M.I.T. Press* \$4.00.)

This book is one of the M.I.T. Press Research Monograph series and is intended to show recent research findings and trends in this field. After some 15 years of work in coding theory, there has been very little equipment actually designed and constructed on the basis of the theory. This is mainly because such schemes are very costly to implement in hardware. Constant threshold decoding, as presented by this monograph, is a relatively simple and practical method of error correction. The research reported here led to one of the first fully practical methods of correcting errors in data transmission.

Section I covers some of the history and concepts of threshold decoding starting from Shannon's work in 1948. The threshold decoding of Linear Codes is broken down into five sections followed by a summary. The Sections covered are (*a*) Linear Codes and the Decoding Problems, (*b*) Orthogonal Parity Checks, (*c*) Majority Decoding (*d*) *A Posteriori* Probability Decoding (*e*) Threshold Decoding.

Section II covers some of the background for the succeeding Sections and is a brief discussion of convolutional encoding including algebraic properties of such codes, bounds on code quality, and circuits for encoding and parity checking.

Section III deals with convolutional codes for Threshold decoding. The concept of threshold decoding set forth in Section I is now applied to convolutional codes and several codes are constructed. Section IV covers Threshold decoding for binary Codes. Section V concludes the convolutional codes by covering the error probabilities and performance data that can be obtained by using threshold decoding. Several communication channels and the probability of incorrectly decoding the set of first information symbols are discussed at some length.

Sections VI and VII cover Threshold decoding of Block codes in the same manner as Sections IV and V did for binary codes.

Section VIII covers the conclusions of the research and opens up some very interesting areas for future research.

Four Appendices are included covering basic definitions and properties of modern algebra and proofs of some of the Theorems used.

In conclusion this is no book for the layman but is a good buy for the serious worker in coding theory.

R. C. WILLIAMS.