An alternative method of solution of certain tri-diagonal systems of linear equations

By M. D. Bakes*

A technique is described for solving certain tri-diagonal systems of linear equations which is somewhat different from that used by D. J. Evans and C. V. D. Forrington (this *Journal*, January 1963). The class of equations amenable to the present treatment is rather more general than that given in the above paper, and some reduction of calculating effort is obtained.

In a paper in the January 1963 issue of this *Journal* Evans and Forrington presented a method for solving equations of the form

$$\begin{bmatrix} a & b & & & & \\ b & a & b & & & \\ & b & a & b & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & \cdot & \\ & & & & b & a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ x_N \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ d_N \end{bmatrix} \quad (1)$$

under the assumption that $a^2 \geqslant 4b^2$ and $a > 0$. After suitable normalization the operations required to solve equations (1) were $4N$ additions and multiplications. It was not possible to reduce this number when more than one right-hand side was used with the same matrix. The authors noted that if x_1 could be found independently then the number of operations required to solve for the remaining x_i by their method would be $2N(M + A)$, where M and A represent the operations of multiplication and addition. It will be shown in the following that when a number of different right-hand sides are used with the same matrix the value of x_1 can be found after only $N(M + A)$ operations. If the method of Evans and Forrington is used to find the remaining x_i a total of only $3N(M + A)$ operations is required for the complete solution.

The normalization referred to above is effected by dividing each side of the equations (1) by $\frac{1}{2}[a' + \sqrt{(a'^2 - 4b'^2)}]$, where the dashed variables denote the original values of a and b. As a result of the normalization the conditions satisfied by a and b are now

$$a^2 \geqslant 4b^2,\ a > 0, \text{ and } \tfrac{1}{2}[a + \sqrt{(a^2 - 4b^2)}] = 1. \quad (2)$$

This implies that $\sqrt{(a^2 - 4b^2)} = 2 - a$ and therefore $a \leqslant 2$. On squaring both sides of this equation and cancelling common factors

$$b^2 = a - 1. \quad (3)$$

Therefore $0 < b^2 \leqslant 1$, since if $b = 0$ equations (1) are not tri-diagonal.

Equation (1) may be written in matrix notation as

$$\boldsymbol{A}_N\boldsymbol{x} = \boldsymbol{d}. \quad (4)$$

Now let the determinant of $\boldsymbol{A}_N$ be denoted by D_N. By inspection

$$\begin{aligned} D_i &= aD_{i-1} - b^2D_{i-2} \\ &= (1 + b^2)D_{i-1} - b^2D_{i-2} \quad (i = 2, \ldots, N) \end{aligned} \quad (5)$$

where $D_1 = 1 + b^2$, and $D_0 = 1$.

The solution of this equation is

$$\begin{aligned} D_i &= \frac{1 - b^{2+2i}}{1 - b^2} \quad \text{when } 0 < b^2 < 1 \\ &= 1 + i \quad \text{when } b^2 = 1. \end{aligned} \quad (6)$$

Now consider the inverse of $\boldsymbol{A}_N$. The first row of this inverse is

$$\left\{\frac{D_{N-1}}{D_N}, \frac{-bD_{N-2}}{D_N}, \frac{b^2D_{N-3}}{D_N}, \ldots, \frac{(-b)^{i-1}D_{N-i}}{D_N}, \ldots, \frac{(-b)^{N-1}}{D_N}\right\}$$

and therefore

$$x_1 = \sum_{i=1}^{N} \frac{(-b)^{i-1}D_{N-i}d_i}{D_N}. \quad (7)$$

Substituting from equation (6) gives

$$x_1 = \sum_{i=1}^{N} \frac{(-b)^{i-1}(1 - b^{2+2N-2i})d_i}{1 - b^{2+2N}} \quad \text{when } 0 < b^2 < 1, \quad (8)$$

and

$$x_1 = \sum_{i=1}^{N} \frac{(-\operatorname{sign} b)^{i-1}(N + 1 - i)d_i}{N + 1} \quad \text{when } b^2 = 1. \quad (9)$$

When b^2 is almost equal to 1 the use of equation (8) would introduce inaccuracies due to the calculation of terms like $1 - b^{2+2N-2i}$. Equations (10) and (11) below both avoid this difficulty, although (10) will be much less accurate than (11), since the difference between two nearly equal numbers will still be involved.

If there is only one set of equations (1) with a given matrix $\boldsymbol{A}_N$, equation (8) can be written in the form

$$x_1 = \sum_{i=1}^{N} \frac{(-b)^{i-1}(d_i - Bd_{N+1-i})}{1 - B^2} \quad \text{when } 0 < b^2 < 1 \quad (10)$$

* *Operational Research Department, Central Electricity Generating Board, North Eastern Region,* 1, *Whitehall Road, Leeds,* 1.

where $B = (-b)^{N+1}$. Using nested multiplication the evaluation of x_1 from this equation requires $2N(M + A)$ operations and the evaluation of B, which is comparable with the method of Evans and Forrington. In fact, in terms of programming effort and accuracy it is better to find x_1 as in the latter method when $0 < b^2 < 1$. However, when $b^2 = 1$, the evaluation of x_1 from equation (9) takes only $N(M + A)$ operations.

If theie are two or more sets of equations (1) with a given matrix A_N, equation (8) can be written in the form

$$x_1 = \sum_{i=1}^{N} \frac{(-b)^{i-1}(1 + b^2 + \ldots + b^{2N-2i})d_i}{1 + b^2 + \ldots + b^{2N}} \quad \text{when } 0 < b^2 < 1. \tag{11}$$

The values of b^{2i} can be found by successive multiplication of b^{2i-2} by b^2, and $1 + b^2 + \ldots + b^{2N}$ can be found by repeated addition of these values. If the expression $(-b)^{i-1}(1 + b^2 + \ldots + b^{2N-2i})$ is denoted by a_i, then a_i can be found from the recurrence relations

$$\left.\begin{aligned} a_{2j} &= -b(a_{2j-1} - b^{2N-2j}) \\ a_{2j+1} &= -ba_{2j} - b^{2N-2j} \end{aligned}\right\} (j = 1, \ldots, [N/2]) \tag{12}$$

where $a_1 = 1 + b^2 + \ldots + b^{2N-2}$.

The calculation of all the a_i takes $2N(M + A)$ operations, so that if there are P different right-hand sides with a given A_N, the complete solution will take $(2 + 3P)N(M + A)$ operations, compared with $4PN(M + A)$ opeiations using the method of Evans and Forrington.

Now consider the case where $a'^2 < 4b'^2$ and $a' > 0$. If the equations (1) are normalized so that b' is replaced by -1 the equation for D_i is now

$$D_i = aD_{i-1} - D_{i-2} \qquad (i = 2, \ldots, N) \tag{13}$$

where $D_1 = a$ and $D_0 = 1$, and the equation for x_1 is

$$x_1 = \sum_{i=1}^{N} \frac{D_{N-i}d_i}{D_N}. \tag{14}$$

Since $|a| < 2$, the recurrence relation (13) provides a reasonably stable means of calculating the D_i (N.P.L., 1961), needing approximately $N(M + A)$ operations. Using these values of D_i in equation (14) involves a further $N(M + A)$ operations to find x_1.

It is not possible to use the method of Evans and Forrington to find the remaining x_i, since it is unstable when $a^2 < 4b^2$. However, the normalized equations (1) can be written as

$$x_{i+1} = ax_i - x_{i-1} - d_i \qquad (i = 1, \ldots, N - 1) \tag{15}$$

where $x_0 = 0$, and the stability properties of equation (15) are the same as those of equation (13). The evaluation of the remaining x_i from equation (15) requires approximately $N(M + 2A)$ operations.

If A_N appears with only one right-hand side the complete solution will require $N(3M + 4A)$ operations, and if A_N appears with P right-hand sides the complete solution will require $N(2P + 1)$ multiplications and $N(3P + 1)$ additions.

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References

EVANS, D. J., and FORRINGTON, C. V. D. (1963). "Note on the solution of certain tri-diagonal systems of linear equations," *The Computer Journal*, Vol. 5, p. 327.

N.P.L. (1961). *Modern Computing Methods*, 2nd edition, London: H.M.S.O.

Correspondence

To the Editor,

The Computer Journal.

Sir,

With reference to my article "The numerical solution of second-order differential equations not containing the first derivative explicitly" (*The Computer Journal*, Vol. 6, p. 368), I have been informed that formula (D) given there has also been given by Albrecht (1955). I am grateful to Dr. Albrecht for drawing my attention to this.

Yours faithfully,

R. E. SCRATON.

Northampton College of Advanced Technology,
St. John Street, London E.C.1.

Reference

ALBRECHT, J. (1955). "Beiträge zum Runge–Kutta-Verfahren," *Z. angew. Math. Mech.*, Vol. 35, p. 100.