Numerical solution of Fredholm integral equations of first kind

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The solution of Fredholm integral equations of the first kind is considered in terms of a linear combination of eigenfunctions of the kernel. Practical and theoretical difficulties appear when any corresponding eigenvalue is very small, and "partial" solutions are obtained which exclude the "small" eigensolutions and which are exact for a slightly perturbed integral equation. Methods are discussed for simplifying the computation of the relevant eigensolutions, and four numerical examples are treated in detail.

Introduction

1. It is well known that the Fredholm integral equation of the first kind is somewhat difficult to solve. In §§2–5 of this paper we recall in general terms the analytical nature of the problem, and in §§6–7 we show in some detail how this is reflected in standard numerical methods. The problem is essentially ill-conditioned, in the sense that there are many solutions which satisfy exactly an integral equation slightly perturbed from the original, and we might therefore decide to seek a "smooth" solution rather than an exact solution. There are many ways of selecting smoothness, and here we concentrate, with several numerical examples, on finding solutions represented by linear combinations of the eigenfunctions corresponding to the dominant eigenvalues of the kernel, noting that in general the contributions from functions belonging to very small eigenvalues are highly oscillatory and difficult to obtain accurately.

Analytical difficulties

2. The Fredholm equation of second kind, given by

$$\int_a^b k(x, y)f(y)dy = \lambda f(x) + g(x), \tag{1}$$

can be solved for a non-singular kernel $k(x, y)$ by expressing the integral in (1) in terms of a finite-difference or Gauss-type quadrature formula, and obtaining approximate "pivotal" values of $f(y)$ from a set of linear simultaneous algebraic equations. For example we might use a Newton–Cotes-type integration formula to obtain for (1) the representation

$$\int_a^b k(x_s, y)f(y)dy = h \sum_{r=1}^{n} \{w_r k(x_s, y_r)f(y_r)\} + e_s = \lambda f(x_s) + g(x_s), \tag{2}$$

where e_s is the "error term" in the quadrature formula, depending on the functions k, f and also on x_s, h is the constant interval between pivotal points, and w_r is the weighting coefficient at pivotal point y_r. The aggregate of these equations for the pivotal points $x_s = y_1, y_2, \ldots, y_n$ then produces algebraic equations represented by

$$hKDf = \lambda f + g - e, \tag{3}$$

where K, the "matrix of the kernel," is given in obvious notation by

$$K = \begin{bmatrix} k_{11} & k_{12} & \ldots & k_{1n} \\ k_{21} & k_{22} & \ldots & k_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ k_{n1} & k_{n2} & \ldots & k_{nn} \end{bmatrix}, \tag{4}$$

the diagonal matrix D is $\mathrm{diag}(w_1, w_2, \ldots, w_n)$, and f, g and e are vectors whose components are the respective pivotal values f_s, g_s, e_s, for $s = 1, 2, \ldots, n$.

For Fredholm equations of the first kind the coefficient λ in (1) is zero, and the corresponding equations of type (3) become

$$hKDf = g - e. \tag{5}$$

We note in passing that the values of x_s need not here be the same as the y_r, though their number must be the same in order to produce a square matrix K, and if $k(x, y) = k(y, x)$ we preserve symmetry in the algebraic problem if we take the x_s to be the pivotal points of the quadrature formula.

3. In general practice we attempt to choose an interval h for which the error vector e and its effect on the solution are negligible, and for this purpose we commonly obtain and compare successive solutions for different values of the interval h in (2) or for different numbers of points in the corresponding Gauss formula.

Alternatively (Fox and Goodwin, 1953; Fox, 1962) we might retain the same interval, but include the effect of the error term in an iterative process typified for (3) by

$$hKDf^{(r+1)} = \lambda f^{(r+1)} + g - e^{(r)}, \quad e^{(0)} = 0, \tag{6}$$

the iteration being necessary because e depends on the yet unknown f.

It is clear that the success of either method depends on the behaviour of the matrix $(\lambda I - hKD)^{-1}$. For the first method, for example, suppose that the true f, the solution of (1), satisfies equations (3) while the computed F satisfies

$$hKDF = \lambda F + g, \tag{7}$$

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with e neglected. The error vector $\eta = f - F$ satisfies at the pivotal points the equation

$$\eta = (\lambda I - hKD)^{-1}e, \tag{8}$$

and for convergence of the computed solution to the true solution we must not only have $e \to 0$ with h, which is necessary anyway for the success of the quadrature formula in (2), but also $(\lambda I - hKD)^{-1}e$ must tend to the null vector as $h \to 0$. For this purpose the inverse must exist, that is λ must not be a limit point of the eigenvalues of the matrices hKD as $h \to 0$.

4. This is reflected in the mathematical theory of integral equations. If $e(x)$ is the error in the quadrature formula the error function

$$\eta(x) = f(x) - F(x) \tag{9}$$

itself satisfies the integral equation

$$\int_a^b k(x, y)\eta(y)dy = \lambda\eta(x) - e(x), \tag{10}$$

and if $\lambda \neq 0$ the conditions for the existence and uniqueness of solutions of (10) are the same as those for solutions of (1).

If there are non-trivial functions $f_r(x)$, corresponding to constants λ_r, such that

$$\int_a^b k(x, y)f(y)dy = \lambda f(x) \tag{11}$$

for $f(x) = f_r(x)$, $\lambda = \lambda_r$, these being the "eigensolutions of the kernel," it is clear that if λ in (1) coincides with any λ_r there can be no unique solution of (1), because any multiple of the eigensolution $f_r(x)$ can be added to any particular solution of (1).

If the solution is unique, and $\lambda \neq 0$, we can write

$$\eta(x) = \lambda^{-1}e(x) + \lambda^{-2}\int_a^b R(x, y; \lambda^{-1})e(y)dy, \tag{12}$$

where $R(x, y; \lambda^{-1})$ is the "resolvent kernel." This does not exist if λ is an eigenvalue, and (12) clearly fails if $\lambda = 0$. For other values of λ the resolvent kernel is bounded, so that $\eta(x) \to 0$ as $e(x) \to 0$. Properties of the resolvent kernel, and allied results, are given by Courant and Hilbert (1953).

5. This analysis fails for the case $\lambda = 0$, that of the Fredholm equation of the first kind. But we can also examine this formally in relation to the homogeneous problem (11). Suppose that (11) has at least some non-zero eigenvalues λ_r and corresponding non-trivial solutions $f_r(x)$, and suppose that we can express $g(x)$ as a linear combination of these eigenfunctions in the form

$$g(x) = a_1 f_1(x) + a_2 f_2(x) + \ldots. \tag{13}$$

Then the solution of (1), with $\lambda = 0$, is given by

$$f(x) = \frac{a_1}{\lambda_1}f_1(x) + \frac{a_2}{\lambda_2}f_2(x) + \ldots. \tag{14}$$

For analytic success both series (13) and (14) must converge, and without going deeply into the theory it is clear that we might fail if $\lambda_r \to 0$ too rapidly. Unfortunately, this is only too likely. If the kernel is degenerate, that is of the form

$$k(x, y) = \sum_{r=1}^{p} X_r(x)Y_r(y), \tag{15}$$

there can be at most only p non-zero eigenvalues. In that event (14) can converge, becoming a finite series, only if (13) is satisfied exactly by a combination of the eigenfunctions belonging to these non-zero eigenvalues. But our solution is not then unique, because we can add to the right of (14) any combination of eigenfunctions belonging to zero eigenvalues.

Another possibility is that the kernel has an infinity of non-zero eigenvalues. Compared, however, with the self-adjoint differential-equation problem of type

$$y'' + \lambda y = 0, \tag{16}$$

which has an infinity of eigenvalues with infinity as limit point, those of the kernel crowd into the origin, with obvious disastrous effect.

Numerical difficulties

6. These situations have close analogies in the numerical solution of the corresponding algebraic problem. Consider first the degenerate kernel (15). Here the corresponding matrix K of (4) is singular if its order n exceeds p, and this is quite independent of the nature of the functions $X(x)$, $Y(y)$ in (15) (Fox, 1962). In particular these functions need not be polynomials, or even have polynomial-like behaviour which was suggested by Fox and Goodwin (1953) as the main reason for the difficulty. Now we may want to take more than p points in the quadrature formula just to make the error term satisfactorily small, and in this sense, of course, the Gauss quadrature has obvious advantages. But with a singular matrix, and even with a $g(x)$ which permits a solution, we do not have a unique solution and must make some decision about what solution is acceptable.

Consider, for example, the equation

$$\int_0^1 (x + y)f(y)dy = g(x), \tag{17}$$

with degenerate kernel. The two non-zero eigenvalues and corresponding eigenfunctions are

$$\lambda_1, \lambda_2 = \tfrac{1}{2} \pm 3^{-1/2}; \quad f_1(x), f_2(x) = x \pm 3^{-1/2}, \tag{18}$$

and we can find no solution unless $g(x)$ is also linear.

The use of the trapezium rule, with the points $x_s = 0$ and 1 in (2), gives the equations

$$f(1) = 2g(0), \quad f(0) + 2f(1) = 2g(1), \tag{19}$$

with a perfectly definite but generally useless solution for arbitrary $g(x)$. With Simpson's rule, or with more points in the trapezium rule, the matrix is singular.

For example, with Simpson and $x_s = 0, \frac{1}{2}$ and 1 we find

$$\left.\begin{aligned} 2f(\tfrac{1}{2}) + f(1) &= 6g(0) \\ \tfrac{1}{2}f(0) + 4f(\tfrac{1}{2}) + \tfrac{3}{2}f(1) &= 6g(\tfrac{1}{2}) \\ f(0) + 6f(\tfrac{1}{2}) + 2f(1) &= 6g(1) \end{aligned}\right\}. \tag{20}$$

We deduce that there is no solution unless $g(\frac{1}{2}) = \frac{1}{2}(g(0) + g(1))$, confirming the known necessity for linearity of $g(x)$. The resulting solution, however, can be written as

$$f(0) = 6g(1) - 18g(0) + 2a,$$
$$f(\tfrac{1}{2}) = 3g(0) - a, \quad f(1) = 2a, \tag{21}$$

for any value of the parameter a.

With four points in the quadrature formula we have two arbitrary parameters, and so on, corresponding to the fact that the true solution can contain any multiple of any $f(x)$ which satisfies

$$\int_0^1 f(y)dy = 0, \quad \int_0^1 yf(y)dy = 0, \tag{22}$$

corresponding to zero eigenvalues of the kernel.

7. When the kernel is not degenerate it would seem that we may have some chance of finding a solution, and even a unique solution. What happens in general practice is that, in the solution of the linear equations of type

$$hKDf = g, \tag{23}$$

which is (5) with the error term neglected, we obtain fairly smooth solutions with a small number of pivotal points. They are likely to be inaccurate, however, because the error term of the quadrature formula is not in fact small enough. As the order of the matrix increases, with more points in the quadrature, the results show increasingly divergent oscillation.

The reasons for this are fairly obvious. First, the more points we take the more accurately can our kernel be approximated by the degenerate form (15), so that our matrix becomes increasingly nearly singular. Second, the algebraic equivalent of the solution of (23), corresponding to equations (13) and (14), is

$$\left.\begin{aligned} g &= a_1 f^{(1)} + a_2 f^{(2)} + \ldots + a_n f^{(n)} \\ f &= \frac{a_1}{\lambda_1} f^{(1)} + \frac{a_2}{\lambda_2} f^{(2)} + \ldots + \frac{a_n}{\lambda_n} f^{(n)} \end{aligned}\right\}, \tag{24}$$

where $f^{(r)}$ is an eigenvector, λ_r the corresponding eigenvalue, of the matrix hKD. If the matrix is of order n, with distinct non-zero eigenvalues and therefore a full set of vectors, equations (24) are certainly possible and give the exact solution of the algebraic problem. Now with n points we would expect to get those eigenfunctions, or at least approximations to them, for which the error term is small, and that means the eigenfunctions with fewest changes of sign, the smoothest in this sense. Moreover, unlike the differential-equation problem, these are likely to be associated with the larger eigenvalues. As we decrease the interval we get better approximations to these eigensolutions, but we also introduce approximations to more and more of the highly oscillating functions associated with the small eigenvalues. These approximations, moreover, are rather poor, and high relative errors in small λ give large absolute errors in the second of (24).

All this, in turn, is associated with the fact that the solution of (23) can be written in the form

$$f = (hD)^{-1}(K^{-1}g). \tag{25}$$

For the trapezium rule $D = \mathrm{diag}(\frac{1}{2}, 1, 1, \ldots, 1, \frac{1}{2})$, while for any rule with the same pivotal points but with different weights we have $D = \mathrm{diag}(w_1, w_2, \ldots, w_n)$, and the solution can be obtained immediately from that for the trapezium rule by multiplying the successive components by the factors $\frac{1}{2}w_1^{-1}, w_2^{-1}, w_3^{-1}, \ldots, \frac{1}{2}w_n^{-1}$. This phenomenon, an example of which appears in Table 4 of §17, is independent of the functions appearing in the integral equation and of the size of the error terms in the quadrature formulae.

Approximate solutions

8. The analysis suggests that we might best try to get an *approximate* solution to the given integral equation, our solution being *exact* for a slightly perturbed equation

$$\int_a^b k(x, y)f(y)dy = g(x) + \varepsilon(x), \tag{26}$$

by suppressing in the expansions (24) the contributions from the smaller eigenvalues and corresponding functions. Our success will depend mainly on the accuracy with which $g(x)$ can be fitted with the first few terms of the first of (24), and in this respect we may get a reasonably smooth solution of (26), which might have value in physical problems in which $g(x)$ has errors of measurement, when $\varepsilon(x)$ is less than the "tolerance."

This suggestion represents some kind of smoothing process, different in kind from that suggested by Fox (1962) for degenerate kernels, in which the parameters of type a in (21) were adjusted to give smooth differences of $f(x)$, or from that of Phillips (1962) who implicitly chooses that $\varepsilon(x)$ in (26) which minimises $\int_a^b (f''(x))^2 dx$.

9. With the use of eigenfunctions (the suggestion of which is not new in principle) we do not mind reducing the interval, since we are concerned only with improving the approximation to the first few dominant eigensolutions, and we have perfectly good methods for finding these solutions in order of decreasing eigenvalues.

In particular, if the kernel is symmetric so that $k(x, y) = k(y, x)$, a case of considerable importance, we can take corresponding advantage in the algebraic problem by replacing (23) by

$$C\phi = \psi, \quad C = hD^{1/2}KD^{1/2}, \quad \phi = D^{1/2}f, \quad \psi = D^{1/2}g. \tag{27}$$

The eigenvalues of the symmetric matrix C are those of hKD, and if we expand ψ in terms of the eigenvectors $\phi^{(r)}$ of C we have

$$\psi = \Sigma\alpha_r\phi^{(r)}, \quad \alpha_r = \phi^{(r)\prime}\psi/\phi^{(r)\prime}\phi^{(r)}, \tag{28}$$

in virtue of the orthogonality of the eigenvectors, and the solution is given by

$$f = D^{-1/2}\phi, \quad \phi = \Sigma\left(\frac{\alpha_r}{\lambda_r}\right)\phi^{(r)}. \tag{29}$$

Example 1

10. As a simple but instructive example we consider first the integral equation (17) whose kernel is degenerate. The trapezium rule will not give the exact solution at any interval, but the corresponding matrix hKD will always have only two non-zero eigenvalues, and these eigenvalues and corresponding eigenfunctions will approximate more closely, as h decreases, to those of the kernel of the integral equation. The latter are given by (18), and for the case $g(x) = x$ the exact linear solution is $f(x) = 4 - 6x$.

11. The calculation of the eigensolutions of any degenerate kernel is facilitated by the observation that if $k(x, y)$ is given by (15), and we use the quadrature formula involved in (2), then the (i, j) element of the matrix hKD is given by

$$(hKD)_{ij} = hw_j \sum_{r=1}^{p} X_r(x_i)Y_r(y_j). \tag{30}$$

The matrix hKD has rank p, and we can express it as a product AB of matrices of respective shapes $(n \times p)$ and $(p \times n)$, for $p < n$, where

$$(A)_{ij} = X_j(x_i), \quad (B)_{ij} = hw_jY_i(y_j). \tag{31}$$

The non-zero eigenvalues of hKD are then those of the $(p \times p)$ matrix BA, and if ϕ is an eigenvector of BA the corresponding vector of hKD is $A\phi$.

If we take $x_i = y_i$ it follows in the general case that

$$(BA)_{ij} = \text{Comp.}\int_a^b Y_i(y)X_j(y)dy, \tag{32}$$

where the right-hand side of (32) means the result of applying our quadrature formula to evaluate the integral. This process, with the integrals in (32) evaluated *exactly*, gives the *exact* eigensolutions of the kernel.

12. In our example $k(x, y) = x + y$, so that

$$p = 2, \quad X_1 = x, \quad Y_1 = 1, \quad X_2 = 1, \quad Y_2 = y, \quad (a, b) = (0, 1), \tag{33}$$

and for the trapezium rule we find

$$BA = \begin{bmatrix} \frac{1}{2} & 1 \\ \frac{1}{3}(1 + \frac{1}{2}h^2) & \frac{1}{2} \end{bmatrix}, \tag{34}$$

the term in h^2 arising from the trapezium-rule approximation to $\int_0^1 y^2dy$, the other approximations being exact for the integration of a constant and a linear function.

We find the eigenvalues

$$\lambda_1, \lambda_2 = \tfrac{1}{2} \pm 3^{-1/2}(1 + \tfrac{1}{2}h^2)^{1/2}, \tag{35}$$

and corresponding eigenvectors with components 1, $\pm 3^{-1/2}(1 + \frac{1}{2}h^2)^{1/2}$. The corresponding vectors of hKD have rth components equal to $rh \pm 3^{-1/2}(1+\frac{1}{2}h^2)^{1/2}$, and comparison with (18) shows in passing that our trapezium-rule approximations have errors represented by a series of powers of h^2, verifying our expectation of the validity of "h^2-extrapolation."

We may also note in passing that the linearity of the finite-difference eigenvectors can also be proved by noticing that the eigensolutions come from the equations

$$h\sum_{r=1}^{n} w_r(x + y_r)f(y_r) = \lambda f(x), \tag{36}$$

in which x is given the specific values y_r. But the computation of $f(x)$ for any x can then be effected from (36), for $\lambda \neq 0$, so that we can regard x in (36) as a continuous variable. Differentiation of (36) with respect to x then gives

$$h\sum_{r=1}^{n} w_rf(y_r) = \lambda f'(x), \tag{37}$$

so that $f'(x)$ is constant and $f(x)$ is linear in x.

13. We need rather a small interval to get a very good approximation to the true solution. For example with $h = \frac{1}{4}$, and α_1 and α_2 defined in (28), we find

$$\left.\begin{array}{l} \lambda_1 = 1{\cdot}086302, \ \lambda_2 = -0{\cdot}086302, \\ \alpha_1\lambda_1^{-1} = 1{\cdot}0390, \quad \alpha_2\lambda_2^{-1} = -3{\cdot}6861 \\ f = (3{\cdot}667, 2{\cdot}333, 1{\cdot}000, -0{\cdot}333, -1{\cdot}667) \end{array}\right\}, \tag{38}$$

with significant errors compared with $4 - 6x$. This is due largely to the *relative* error in λ_2, which is about 10%, and the fact that the contribution from its eigenvector has a substantial factor.

Simpson's rule, of course, gives the exact result at any interval.

Example 2

14. As a second example we treat the equation

$$\int_0^1 (x^2 + y^2)^{1/2}f(y)dy = \tfrac{1}{3}\{(1 + x^2)^{3/2} - x^3\}, \tag{39}$$

whose solution is $f(x) = x$. This was considered by Fox and Goodwin (1953), who noted the oscillation produced by the direct solution of the trapezium-rule finite-difference equations in the table

x	0	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{3}{8}$	$\frac{4}{8}$
f	0·0144	0·1464	0·1573	0·6822	−0·2338

$\frac{5}{8}$	$\frac{6}{8}$	$\frac{7}{8}$	1
1·8157	−0·5076	1·7323	0·4003.

(40)

Here the kernel is not degenerate, and there is an infinity of non-zero eigenvalues. We find them in decreasing order of magnitude, and compute solutions from (29) expressed in the form

$$f = D^{-1/2} \sum_{r=1}^{s} \alpha_r \lambda_r^{-1} \phi^{(r)}, \tag{41}$$

taking $s = 1, 2, \ldots$, and choosing its appropriate terminal value. Some results are given in Table 1, for interval $h = \frac{1}{4}$ with the use of the trapezium rule. For simplicity we tabulate in each case the error, rather than the computed solution, since here we know the true result.

Table 1 ($h = \frac{1}{4}$)

$4x$	$s=1$	$s=2$	$s=3$	$s=4$	$s=5$
0	−0·370	−0·0956	−0·0123	−0·0399	−0·0299
1	−0·168	−0·0056	−0·0415	0·0185	−0·0332
2	−0·001	0·0198	−0·0146	−0·0674	0·1010
3	0·118	0·0280	0·0287	−0·0019	−0·2080
4	0·235	0·0315	0·0752	0·1440	0·3110

$\lambda_1 = 0{\cdot}8227$, $\lambda_2 = -0{\cdot}1028$, $\lambda_3 = -0{\cdot}0113$, $\lambda_4 = -0{\cdot}0015$, $\lambda_5 = 0{\cdot}0001$

The results are best for $s = 3$, though little inferior for $s = 2$, and in the latter case the vector $g - a_1 f_1 - a_2 f_2$ has a maximum component of about 0·0009.

15. But we must be concerned about the error term in the quadrature formula. If this is not negligible our algebraic eigenfunctions are not quite those of the kernel, and will in fact contain small multiples of some of the unwanted eigenfunctions of the true problem. We might therefore decide that we can get a reasonable fit with say three eigenfunctions, and then proceed to get very accurately the first three eigensolutions. The trapezium rule is rather inefficient for this purpose, so that we repeat the computation with Simpson's rule, taking small enough intervals to justify faith in both λ_r and $\phi^{(r)}$, $r = 1, 2, 3$. We also give the error in the results corresponding to the use of just one and two accurate eigenfunctions, respectively, and all these are shown in Table 2. The accuracy even in the fourth decimal of λ_r cannot be guaranteed without taking h as small as 1/16, but we show the conclusions only at the pivotal points $x = 0(\frac{1}{4})1$.

Table 2 ($h = \frac{1}{16}$)

$4x$	$s=1$	$s=2$	$s=3$
0	−0·375	−0·0725	−0·0255
1	−0·172	−0·0155	−0·0031
2	−0·019	0·0151	−0·0032
3	0·108	−0·0012	−0·0011
4	0·220	−0·0204	−0·0046

$\lambda_1 = 0{\cdot}81085$, $\lambda_2 = -0{\cdot}09565$, $\lambda_3 = -0{\cdot}00655$

Example 3

16. For the kernel e^{xy} in the range (0, 1) the eigenvalues get small even more rapidly, and in Table 3 we show the first four, obtained by Gauss quadrature with various numbers N of points.

Table 3

N	λ_1	λ_2	λ_3	λ_4
2	1·35208	0·10209		
4	1·35303	0·10598	0·00356	0·00007
6	1·35303	0·10598	0·00356	0·00008
8	1·35303	0·10598	0·00356	0·00008

Here we should clearly expect to get a reasonable solution with at most the first three eigenfunctions, and indeed it would be dangerous to use the fourth unless we compute it to many more figures. In fact for the equation

$$\int_0^1 e^{xy} f(y)dy = (x+1)^{-1}(e^{x+1} - 1), \tag{42}$$

whose solution is e^x, we find with $N = 8$ calculations whose maximum errors are 0·36, 0·017, and 0·0004 corresponding to the use of λ_1, λ_1 and λ_2, and λ_1, λ_2 and λ_3. With more eigenfunctions the error becomes greater, and with six it has a maximum of as much as 1·2.

Somewhat surprisingly we get an extremely accurate result by solving the linear equations directly with $N = 4$, this corresponding, of course, to the use of all four relevant eigensolutions. The error is only about 0·0001, and we conclude that the contribution from the function corresponding to λ_4 is quite small and unaffected in the fourth decimal by a probably large relative error in λ_4. This circumstance, of course, depends on the right-hand side $g(x)$ in the integral equation, and the direct solution of the linear equations with a different $g(x)$ could produce very large errors.

Example 4

17. As a final example we consider the problem discussed by Phillips (1962), given by

$$\left.\begin{aligned} &\int_{-6}^{6} K(x-y)f(y)dy = g(x) \\ &K(z) = 1 + \cos\tfrac{1}{3}\pi z,\ |z| \leqslant 3;\ = 0,\ |z| > 3 \\ &g(x) = (6-|x|)(1 + \tfrac{1}{2}\cos\tfrac{1}{3}\pi x) + \frac{9}{2\pi}\operatorname{sign}(x)\sin\frac{\pi x}{3} \\ &\operatorname{sign} x = 1,\ x \geqslant 0;\ = -1,\ x < 0, \end{aligned}\right\} \tag{43}$$

and whose solution is

$$f(x) = K(x). \tag{44}$$

The discontinuity in the kernel suggests the use of the trapezium rule for the quadrature, and at the interval $h = 1$, giving a matrix of order thirteen, we find eigen-

values ranging in modulus from about 5·8 to 0·008, so that the linear equations are not very badly conditioned. In fact direct solution of these equations gives results summarized in the first row of Table 4, with reasonably small errors. The Simpson-rule results, shown in the second row, are considerably poorer. The results of the first and second rows of Table 4, incidentally, confirm the comment of §7 and equation (25).

Table 4

x	0	±1	±2	±3	±4	±5	±6
Trapezium	1·962	1·539	0·462	0·019	0	0	0
Simpson	2·943	1·154	0·694	0·014	0	0	0
True	2·000	1·500	0·500	0	0	0	0

In fact the right-hand side can be fitted reasonably well with a combination of the first seven eigenfunctions, and the errors in the solution at the integer values of x are shown in Table 5, together with similar results, for comparison, using different numbers of eigenfunctions. The trapezium rule is used in all cases.

Table 5

x	0	±1	±2	±3	±4	±5	±6
$s = 3$	0·445	0·137	−0·371	−0·293	0·134	0·281	0·209
$s = 5$	0·103	0·009	−0·124	0·018	0·124	−0·055	−0·155
$s = 7$	−0·011	0·014	−0·010	0·007	0·009	−0·031	0·045
$s = 10$	−0·016	−0·014	0·009	0·010	−0·023	0·012	−0·007

In practice, of course, we do not know how accurate our results are because we do not know the true solution, so that no reliance can be placed on the conclusions of Table 4. As before we should examine from the results of Table 5 how accurately our function g can be fitted by a combination of the larger eigenfunctions, and then proceed to compute these eigensolutions more accurately with smaller intervals. With $s = 7$ the maximum component of $g - \sum_{r=1}^{7} a_r f^{(r)}$ is about 0·007 at the interval $h = 1$.

Computation of eigensolutions

18. In most of the examples we have computed the eigensolutions by standard processes such as those of Givens for symmetric matrices. We noted in §11, however, some simplification for a degenerate kernel. A matrix C of order n and rank p can be expressed as the product of two rectangular matrices. In particular if the leading submatrices of C of orders 1 to p are non-singular we can write

$$C = AB = \begin{bmatrix} L_1 \\ \hline Q_1 \end{bmatrix} [U_1 | Q_2], \tag{45}$$

where L_1 and U_1 are lower and upper triangles of order p, Q_1 is $(n - p \times p)$, and Q_2 is $(p \times n - p)$. The decomposition can be performed by a standard Gauss elimination process, and Fox (1964) gives some examples of this. We may, however, need to rearrange the rows and columns of C, that is to use "complete pivoting," and the decomposition then refers to $I_r C I_s$, where I_r and I_s are row and column permuting matrices.

If C is symmetric we do well to take pivots on the diagonal, so that $I_s = I_r' = I_r^{-1}$, and $I_r C I_r'$ is an orthogonal similarity transformation of C.

In particular if C is symmetric and positive definite we can use the Cholesky decomposition, replacing (45) by

$$I_r C I_r' = AB = \begin{bmatrix} L_1 \\ \hline Q_1 \end{bmatrix} [L_1' | Q_1'], \tag{46}$$

and the matrix BA is the real symmetric $(p \times p)$ matrix $L_1'L_1 + Q_1'Q_1$.

19. When the kernel is not degenerate its matrix may not be singular, but we have noted that a large matrix, corresponding to the use of many pivotal points, is likely to have several very small eigenvalues. In that case we cannot produce exactly the decomposition (46), but we find in the elimination that all the later elements of some reduced matrix are very small. Moreover, a slight perturbation of the elements of C could make them all zero, and we can say that we have the exact rectangular decomposition for a slightly different kernel.

Alternatively, and particularly in the positive-definite case when all the eigenvalues are real and positive, we note that the sum of the eigenvalues is the trace of the matrix $L_1'L_1 + Q_1'Q_1$, and we can reasonably stop the elimination when there is no significant addition to the sum of squares of the elements in matrix Q_1.

For example, the symmetric matrix $hD^{1/2}KD^{1/2}$, corresponding to the kernel e^{xy} and the trapezium rule at the points 0(0·2)1·0, has for its upper triangular half the array

$$\begin{bmatrix} 0{\cdot}10000 & 0{\cdot}14142 & 0{\cdot}14142 & 0{\cdot}14142 & 0{\cdot}14142 & 0{\cdot}10000 \\ & 0{\cdot}20816 & 0{\cdot}21666 & 0{\cdot}22550 & 0{\cdot}23470 & 0{\cdot}17273 \\ & & 0{\cdot}23470 & 0{\cdot}25425 & 0{\cdot}27543 & 0{\cdot}21098 \\ & & & 0{\cdot}28667 & 0{\cdot}32321 & 0{\cdot}25769 \\ & & & & 0{\cdot}37930 & 0{\cdot}31474 \\ & & & & & 0{\cdot}27183 \end{bmatrix}, \tag{47}$$

and the matrix $I_r C I_r'$, where I_r is the unit matrix with rows permuted in order 5, 2, 6, 4, 1, 3, is very nearly equal to the symmetric rectangular decomposition (46) with

$$[L_1'|Q_1'] = \left[\begin{array}{ccc|ccc} 0{\cdot}61587 & 0{\cdot}38109 & 0{\cdot}51105 & 0{\cdot}52481 & 0{\cdot}22963 & 0{\cdot}44721 \\ & 0{\cdot}25086 & -0{\cdot}08779 & 0{\cdot}10165 & 0{\cdot}21491 & 0{\cdot}18428 \\ & & 0{\cdot}05433 & -0{\cdot}02932 & 0{\cdot}02789 & -0{\cdot}02556 \end{array}\right], \tag{48}$$

the maximum amount of the perturbation being about 0·00031 in the (1, 1) element of (47). The first three eigenvalues of C are respectively 1·36027, 0·11531, and 0·00493, and the eigenvalues of

$$L_1'L_1 + Q_1'Q_1 = \begin{bmatrix} 1{\cdot}31385 & 0{\cdot}23584 & 0{\cdot}00731 \\ 0{\cdot}23584 & 0{\cdot}16111 & -0{\cdot}00649 \\ 0{\cdot}00731 & -0{\cdot}00649 & 0{\cdot}00525 \end{bmatrix} \tag{49}$$

are 1·36026, 0·11528, and 0·00467.

The vectors y of (49) give approximations $I_r'\begin{bmatrix} L_1 \\ \overline{Q_1} \end{bmatrix} y$ to those of C, and their discrepancy, as one might expect, is roughly proportional to the relative error of the corresponding eigenvalues, being very small for the first two and rather large for the third.

Conclusion

20. There are other methods of producing smooth functions which satisfy exactly a perturbed integral equation. Some workers, for example, have assumed a solution of the form

$$f = \Sigma a_r f_r(x), \tag{50}$$

choosing the functions $f_r(x)$ so that the integration can be performed analytically or by numerical quadrature with a small error term. The constants a_r are then computed from a set of linear equations so that the integral equation is satisfied exactly at some points (collocation) or in a least squares sense over the whole range.

One problem here is that the equations for the a_r tend to be ill-conditioned. But we have also to specify in advance the functions $f_r(x)$. Our choice of eigenfunctions of the kernel would seem to be more natural and relevant than any alternative somewhat arbitrary choice.

21. In some cases, however, we may know from other considerations that a particular choice of functions will produce a good result. This happens, for example, for the integral equation

$$\left.\begin{aligned} \int_0^1 k(x, y)f(y)dy &= \tfrac{1}{6}x(1 - x^2), \quad 0 \leqslant x \leqslant 1 \\ k(x, y) &= (1 - x)y, \quad 0 \leqslant y \leqslant x \leqslant 1 \\ &= (1 - y)x, \quad 0 \leqslant x \leqslant y \leqslant 1 \end{aligned}\right\}, \tag{51}$$

which has the continuous solution $f(x) = x$ in $0 \leqslant x \leqslant 1$. This problem was discussed by Tricomi (1957), who showed that the eigensolutions of the kernel are

$$\lambda_r = (r\pi)^{-2}, \quad f_r(x) = \sin(r\pi x). \tag{52}$$

It is clear that no combination of the eigenfunctions can produce exactly the solution $f(x) = x$ at $x = 1$, since $\sin(r\pi x)$ vanishes at this point. We have the type of point-wise error which is peculiar to a Fourier series of sine terms. The cosine series can give a good approximation at every point, and the substitution

$$f(y) = \Sigma a_r \cos(r\pi y) \tag{53}$$

in (51) leads to a set of equations for the a_r which, at least up to $r = 20$, are not particularly ill-conditioned.

22. The numerical solution of (51) by our methods shows some interesting points. Since the kernel has a discontinuous y-derivative at $x = y$ we prefer the trapezium rule for the quadrature, and the matrix hKD has zero elements in its first and last rows and columns. This matrix therefore has two zero eigenvalues, whose eigenvectors are the first and last columns of the unit matrix. The eigenvectors corresponding to non-zero eigenvalues have zeros in the first and last components. Moreover, though the trapezium-rule matrix hKD does not give the true eigenvalues at a finite interval it does give the true eigenfunctions, and the direct solution of the linear equations at any interval gives $f(x) = x$ at all points except $x = 1$.

23. Our examples all involve symmetric kernels, and we hope to continue experimenting with more difficult problems, for example, those in which eigenfunctions of large oscillation may be associated with relatively large eigenvalues. The unsymmetric case will necessitate a different method of choosing the constants α_r in (28), and even that choice in the symmetric case, which minimizes the sum of squares of the components of the vector $\psi - \Sigma\alpha_r\phi^{(r)}$, is not necessarily the best in all cases. We might, for example, sometimes prefer to minimize the largest component in absolute value.

Other questions include an evaluation of the method of §19 for finding the eigensolutions, and this also we hope to investigate further.

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Book review: Linear programming

Linear Programming and Extensions, by GEORGE B. DANTZIG, 1963; 625 pages. (Princeton: *University Press*; London: *Oxford University Press*, 92s.)

This long-awaited book by Professor Dantzig will certainly come to be regarded as one of the classic textbooks on linear programming. I can highly recommend it to all interested in the theory of linear programming and its more useful extensions, particularly to those interested in the economic interpretation of linear programming models, and also for use as a library reference book. The treatment of the subject matter is mathematically complete, and explained in sufficient detail to make it as readable as possible. This approach, together with the wealth of material which the book contains, explains its great size. It gives almost comprehensive coverage to the main developments in the subject until the end of 1960.

Just how many years ago it was that I first heard that Dr. Dantzig was writing a book on linear programming I now forget. But as the years passed and it failed to appear, "Dantzig's book" became almost a joke. Now the preface reveals the great organization that went into its preparation. No less than 24 people, most of them well known in linear programming circles, including five professors and nine doctors, are thanked for contributing to the writing of various sections, and a further nine people for helping with the layout, proof-reading and indexing. The result is a book of authority and of technical excellence.

Naturally, with so many hands contributing, the style is not entirely uniform. But this variety adds a certain spice. Some sections in which the ideas are conveyed in story form such as "The Scheme of the Ambitious Industrialist" make particularly good reading, whilst in other chapters the ideas of several papers are rather loosely strung together. An occasional remark reveals the long period of gestation, such as that in the second chapter about developments in the ten years since 1947.

The chapter on the origins and early influences on linear programming is of great interest. But Professor Dantzig is too modest to allow the reader to recognize the full impact which the team under his leadership at the Rand Corporation had on the rapid development of this subject in the nineteen fifties.

Professor Dantzig believes in the usefulness of approaching his subject in many ways in order to gain as many insights into it as possible. The simplex method is therefore explained not only in terms of linear equations and inequalities, but also in terms of matrix algebra, with two distinct geometrical interpretations, economic interpretations and an equivalence with matrix games.

Likewise he covers the transportation problem in a variety of manners: in terms of its own techniques, in terms of the simplex method and in terms of networks and trees. Various special cases, extensions and generalizations of transportation problems and network problems are also discussed.

Extensions to linear programming are treated in five chapters. The first explains the valuable concept of Wolfe's generalized programming scheme. There follows an excellent chapter on the decomposition of linear programs. This includes a short play in which Staff, who hates details, and his economist friend, F. M. Dalks, use the decomposition principle to get Sub to buy the right number of tankers. It also includes a section on the use of the decomposition principle for central planning without complete information at the centre.

A chapter on convex programming includes mention of separable and quadratic programming. There is a chapter on uncertainty, at the start of which the reader is warned that the treatment is necessarily fragmentary as few problems have been solved in this area. Lastly there is a chapter on integer programming which includes a good description of Gomory's method of integer forms but is otherwise mainly a survey of problems which can be put into integer programming form.

In places a number of exercises are suggested to the reader, and many chapters end with a list of problems, a few of which are marked as unsolved.

In his opening sentence Professor Dantzig states firmly that the final test of a theory is its capacity to solve the problems which originated it. Therefore it is surprising to find that he pays so little attention to the implementation of linear programming techniques on digital computers. Whilst there is an elaborate discussion on perturbation techniques which is of great theoretical interest, there is virtually no mention of the perturbation through rounding errors which occurs in all numerical work, nor of the effect which such errors can have on the course of the calculations.

In the detailed iterative procedure given for the two-phase simplex method there is no mention of the tolerance on zero which must be allowed because of rounding errors, and without which the procedure can fail to terminate correctly. The flow diagram given of the simplex method is by no means the most suitable for direct implementation on a computer. The "product form of the inverse" algorithm, which is nowadays most commonly used for large-scale linear programming calculations, receives such a brief mention on page 200 that it is not to be found in either the list of contents or the index. For these reasons the book is not recommended as a handbook for programming linear programming calculations for computers. A good book for that purpose has yet to be written.

MARTIN FIELDHOUSE.