Error estimates for smoothing and extrapolation formulae

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A standard smoothing procedure consists of least squares fitting a polynomial $Q_m(x)$ of degree m to $2n + 1$ observed points and replacing the midpoint (nth point) by the value of the polynomial. In many applications of trajectory extrapolation, or acquisition and reacquisition in radar tracking, it is desirable to smooth at or near the end-point of the observed data. Considering $Q_m(x)$ as a linear combination of a set of orthogonal polynomials $P_{g,2n}(x)$ of degree $g < m$ the error reduction factor for smoothing at an arbitrary jth point is found to be given by

$$f_s(j) = \sum_{g=0}^{m} \frac{P_{g,N}(j)}{S_g}$$

where $S_g = \sum_{x=0}^{N} P^2_{g,N}(x)$ and the total number of observed points, $N + 1$, used in the smoothing may be even or odd. If the jth point belongs to the set of $N + 1$ points, then $f_s(j) = a_{j,j}$, that is, the smoothing factor is equal to the j coefficient of the jth point smoothing formula. Milne's expression for the smoothing factor in midpoint smoothing forms a special case of the above results.

One method of deriving formulae for smoothing equally-spaced data consists of obtaining the least-squares fitted polynomial of degree m through $2n + 1$ successive points, and replacing the function value at the central point with the value of the fitted polynomial. A detailed description of this method is given in Milne's *Numerical Calculus*, together with an estimate of the "smoothing factor," which is an estimate of the improvement achieved by smoothing.

In this paper, the method is generalized by allowing the replacement point to wander from the central position, and expressions are derived for the value of the smoothing factor for arbitrary position of the replacement point. The relations are useful at the ends of a table or for extrapolation problems. End-point smoothing finds important applications in radar tracking of missiles, where a smoothed estimate at the last observation point may be required for reasons of response time or range safety.

Smoothing factors for non-central formulae

Assume a set of $2n + 1$ observations of the function $u(x)$ at equally spaced values of $x (x = 0, 1, 2, \ldots, 2n)$. Denote the true values of u by u_i, the observed values by y_i and the errors in the observed values by e_i. Similarly, denote the smoothed values by y_i' and the errors in the smoothed values by e_i', so that:

$$y_i = u_i + e_i \qquad y_i' = u_i + e_i'.$$

The smoothing factor is then defined by:

$$f_s = \Sigma(e_i')^2/\Sigma(e_i)^2 \tag{1}$$

where Σ denotes summation for fixed i for many repeated observations.

By least-squares fitting a polynomial $Q_m(x)$ of degree m to the $2n + 1$ observed points, the expression for the smoothed central (nth) point may be written as:

$$y_n' = a_{n,0}y_0 + a_{n,1}y_1 + \ldots + a_{n,n}y_n + \ldots a_{n,2n}y_{2n}. \tag{2}$$

Considering the polynomial $Q_m(x)$ to be a linear combination of a set of orthogonal polynomials $P_{g,2n}(x)$ of degree $g \leqslant m$, the coefficients $a_{n,i}$ are given by:

$$a_{n,i} = \sum_{g=0}^{m} \left[\frac{P_{g,2n}(i)P_{g,2n}(n)}{\sum_{x=0}^{2n} P^2_{g,2n}(x)} \right]. \tag{3}$$

Milne showed that for central smoothing f_s is equal to the mid-coefficient of the smoothing formula, i.e.

$$f_s = a_{n,n}. \tag{4}$$

In non-central smoothing or extrapolation we want to find the polynomial value at an arbitrary jth point of a set of $N + 1$ points, where N may be either even or odd.

The value of the polynomial at any such point can be expressed in analogy to (2) and (3) by:

$$Q_m(j) = y_j' = a_{j,0}y_0 + a_{j,1}y_1 + \ldots + a_{j,N}y_N$$

or

$$Q_m(j) = y_j' = \sum_{x=0}^{N} y(x) \sum_{g=0}^{m} \left[\frac{P_{g,N}(x)P_{g,N}(j)}{\sum_{x=0}^{N} P^2_{g,N}(x)} \right]. \tag{5}$$

Since y_j' is a linear combination of the observed y values, then for uncorrelated Gaussian errors in the observations we have:

$$\overline{(e_j')^2} = a^2_{j,0}\overline{(e_0)^2} + a^2_{j,1}\overline{(e_1)^2} + \ldots + a^2_{j,N}\overline{(e_N)^2}$$

and $\overline{(e_k)^2} = \overline{(e_{k+1})^2} = \overline{(e_j)^2}$ so that:

$$f_s(j) = \overline{(e_j')^2}/\overline{(e_j)^2} = \sum_{x=0}^{N} a^2_{j,x}$$

$$= \sum_{x=0}^{N} \left[\sum_{g=0}^{m} \frac{P_{g,N}(x)P_{g,N}(j)}{S_g} \right]^2 \tag{6}$$

where $S_g = \sum_{x=0}^{N} P^2_{g,N}(x)$.

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Expanding the square in (6) we find that it contains terms of the form:

$$\left[\frac{P_{g,N}(x)P_{g,N}(j)}{S_g}\right]^2$$

and terms containing cross products of the form:

$$\frac{P_{g,N}(x)P_{g,N}(j)}{S_g} \cdot \frac{P_{h,N}(x)P_{h,N}(j)}{S_h} \text{ with } h \neq g.$$

Considering the first type of terms, the summation in (6) with respect to x yields:

$$\sum_{x=0}^{N} \sum_{g=0}^{m} \left[\frac{P_{g,N}(x)P_{g,N}(j)}{S_g}\right]^2$$

$$= \sum_{g=0}^{m} \frac{P^2_{g,N}(j)}{S_g^2}. \qquad \sum_{x=0}^{N} P^2_{g,N}(x) = \sum_{g=0}^{m} \frac{P^2_{g,N}(j)}{S_g}.$$

The summation over x of the cross products vanishes because of the orthogonality property of the polynomials $P_{g,N}(x)$, so that (6) reduces to:

$$f_s(j) = \sum_{g=0}^{m} \frac{P^2_{g,N}(j)}{S_g}. \tag{7}$$

For $0 \leqslant i \leqslant N$ we have from (3):

$$f_s(j) = a_{j,j} \tag{8}$$

that is the smoothing factor for jth-point smoothing procedures is equal to the j coefficient in the linear expression of the least-squares fitting polynomial in terms of the observed quantities. Clearly Milne's result (4) is a special case of (8). For smoothing formulae using replacement at points belonging to the observed sequence we find the error reduction in the smoothing process from (8). For extrapolation we have to use relation (7). For end-point smoothing we have the relation:

$$f_s(N) = a_{N,N} = \sum_{g=0}^{m} \frac{P^2_{g,N}(N)}{S_g}. \tag{9}$$

By expansion of the polynomials in (9) and some further manipulations it can be shown* that for mth order smoothing

$$f_s(N)_m \to \frac{(m+1)^2}{N}$$

for large m, while the right-hand side forms an upper bound for smaller m. This same result was obtained using a completely different approach by Proschan (1961).

Numerical examples

For a third-degree central smoothing formula using 21 points we have $f_s(11) = a_{11,11} = 0 \cdot 108$.

For replacement at the 17th point we find:

$$f_s(17) = a_{17,17} = 0 \cdot 374.$$

For the end point: $f_s(21) = a_{21,21} = 0 \cdot 75$.

For smoothed extrapolation at the 22nd and 23rd points

$$f_s(22) = 1 \cdot 13 \text{ and } f_s(23) = 1 \cdot 63.$$

The latter type extrapolations find application in acquisition or reacquisition of targets in radar tracking. We note that beyond the range of the observed data accuracy deteriorates very rapidly.

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