previous section. Since the matrix $\begin{bmatrix} 2Q & A^T \\ A & 0 \end{bmatrix}$ in (4) is symmetric, it is economical to use the square-root method. The number of multiplications is given approximately by

$$\mu' \simeq \tfrac{1}{6}(m+n)^3 = \tfrac{1}{6}(1+\alpha)^3 n^3.$$

The coefficient of n^3 as a function of α is shown by the line labelled (2) in Fig. 1. For α between 0 and 0·5 the amount of computation in the two processes is very nearly the same, but with α in the range 0·5 to 1·0 the procedure given in the previous section becomes considerably more efficient.

A generalized alternating direction method of Douglas–Rachford type for solving the biharmonic equation

By G. Fairweather and A. R. Mitchell*

Alternating direction methods of Douglas–Rachford type are considered as a means of solving the biharmonic equation.
The family of methods obtained contains the Conte–Dames formula as a special case.

Introduction

In the problem of determining the elastic buckling loads of flat plates under partial edge compression, the plate stress u is given by the biharmonic equation

$$\frac{\partial^4 u}{\partial x^4} + 2\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4} = 0. \tag{1}$$

It is convenient to consider the square homogeneous plate $(0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 1)$ supported along its edges and buckled by moments along two opposite plate edges. This leads to the boundary–value problem consisting of (1) together with the boundary conditions

$$\left.\begin{array}{lll} u = \dfrac{\partial^2 u}{\partial x^2} = 0 & \text{at } x = 0, 1 & (0 < y < 1) \\ u = 0 & \text{at } y = 0, 1 & (0 < x < 1) \\ \dfrac{\partial^2 u}{\partial y^2} = f_1(x) & \text{at } y = 0 & (0 < x < 1) \\ \dfrac{\partial^2 u}{\partial y^2} = f_2(x) & \text{at } y = 1. & (0 < x < 1). \end{array}\right\} \tag{2}$$

Several attempts have been made to solve this boundary-value problem using finite differences. Originally, relaxation methods were used, to be followed by more sophisticated methods such as over-relaxation by White (1963) and Tee (1963), and an alternating-direction method by Conte and Dames (1958).

It is the purpose of the present paper to consider alternating direction methods of the Douglas–Rachford (1956) type as a means of solving the biharmonic equation, and to obtain a generalized version of the method used by Conte and Dames.

The Conte Dames (C.D.) method

A square mesh is superimposed over the region $(0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 1)$ with the mesh size $h = 1/N$, where $N(> 3)$ is an integer. The C.D. method for solving (1) is a double-sweep iterative process given by

$$\left.\begin{array}{l} u_{i,j}^{(n+\frac{1}{2})} = u_{i,j}^{(n)} - r(\delta_y^4 u_{i,j}^{(n+\frac{1}{2})} + 2\delta_x^2\delta_y^2 u_{i,j}^{(n)} + \delta_x^4 u_{i,j}^{(n)}) \\ u_{i,j}^{(n+1)} = u_{i,j}^{(n+\frac{1}{2})} - r(\delta_x^4 u_{i,j}^{(n+1)} - \delta_x^4 u_{i,j}^{(n)}), \quad (n \geqslant 0) \end{array}\right\} \tag{3}$$

where $u_{i,j}^{(0)}$ is an initial approximation to u at the node $x = ih$, $y = jh$, δ_x, δ_y denote the usual central difference operators in the x and y directions, respectively, and r is an iteration parameter chosen to accelerate convergence. If $u_{i,j}^{(n+\frac{1}{2})}$ is eliminated, equations (3) become

$$\begin{aligned} u^{(n+1)} = u^{(n)} - r(\delta_x^4 u^{(n+1)} &+ 2\delta_x^2\delta_y^2 u^{(n)} \\ &+ \delta_y^4 u^{(n+1)}) - r^2\delta_x^4\delta_y^4(u^{(n+1)} - u^{(n)}), \end{aligned} \tag{4}$$

where the lower suffices i, j have been omitted.

The C.D. formulae (3) constitute a convergent iterative method for solving the biharmonic equation similar to the Douglas–Rachford alternating-direction method for solving Laplace's equation.

The generalized C.D. formulae

A generalized form of (3) is

$$\left.\begin{array}{r} (a_0 + a_1\delta_y^2 + a_2\delta_y^4)u^{(n+\frac{1}{2})} = (b_0 + b_1\delta_x^2 + b_2\delta_x^4 \\ + b_3\delta_y^2 + b_4\delta_y^4 + b_5\delta_x^2\delta_y^2)u^{(n)} \\ u^{(n+\frac{1}{2})} + (c_0 + c_1\delta_x^2 + c_2\delta_x^4)u^{(n+1)} \\ = (d_0 + d_1\delta_x^2 + d_2\delta_x^4)u^{(n)}. \end{array}\right\} \tag{5}$$

This reduces to the C.D. formulae (3) if the coefficients take the values $a_0 = b_0 = -c_0 = 1$, $a_1 = b_1 = b_3$

* *Department of Applied Mathematics, St. Andrews University, Fife, Scotland.*

$= b_4 = c_1 = d_0 = d_1 = 0, a_2 = - b_2 = - \frac{1}{2}b_5 = - c_2 = - d_2 = r$. If $u^{(n+\frac{1}{2})}u$ is eliminated in (5), the result

$$(a_0 + a_1\delta_y^2 + a_2\delta_y^4)(c_0 + c_1\delta_x^2 + c_2\delta_x^4)u^{(n+1)} = [(a_0 + a_1\delta_y^2 + a_2\delta_y^4)(d_0 + d_1\delta_x^2 + d_2\delta_x^4) - (b_0 + b_1\delta_x^2 + b_2\delta_x^4 + b_3\delta_y^2 + b_4\delta_y^4 + b_5\delta_x^2\delta_y^2)]u^{(n)} \quad (6)$$

is obtained. The coefficients in (6) are now adjusted so that if the process converges, that is $u^{(n+1)} = u^{(n)} = u$, for n sufficiently large, then (6) reduces to

$$\{\alpha_1(\delta_x^4 + 2\delta_x^2\delta_y^2 + \delta_y^4) + \alpha_2\delta_y^2\delta_x^4 + \alpha_3\delta_y^4\delta_x^2 + \alpha_4\delta_y^4\delta_x^4\}u = 0, \quad (7)$$

which is a fourth-order finite-difference replacement of the biharmonic equation for all values of the parameters α_1, α_2, α_3, and α_4. This comes about when the coefficients in (5) and (7) satisfy the relations

$$b_0 = a_0(d_0 - c_0),\ b_1 = -\frac{a_0}{a_2}\alpha_3,\ b_2 = -\frac{a_0}{a_1}\alpha_2 + \alpha_1,$$

$$b_3 = a_1(d_0 - c_0),\ b_4 = a_2(d_0 - c_0) + \alpha_1,$$

$$b_5 = -\frac{a_1}{a_2}\alpha_3 + 2\alpha_1,\ d_1 - c_1 = -\frac{1}{a_2}\alpha_3,$$

$$d_2 - c_2 = -\frac{1}{a_1}\alpha_2 = -\frac{1}{a_2}\alpha_4.$$

In fact using these relations between the coefficients, (6) becomes

$$(a_0 + a_1\delta_y^2 + a_2\delta_y^4)(c_0 + c_1\delta_x^2 + c_2\delta_x^4)u^{(n+1)} = [(a_0 + a_1\delta_y^2 + a_2\delta_y^4)(c_0 + c_1\delta_x^2 + c_2\delta_x^4) - \{\alpha_1(\delta_x^4 + 2\delta_x^2\delta_y^2 + \delta_y^4) + \alpha_2\delta_y^2\delta_x^4 + \alpha_3\delta_y^4\delta_x^2 + \alpha_4\delta_x^4\delta_y^4\}]u^{(n)}. \quad (8)$$

In order to simplify the analysis in what follows and yet to retain the essential character of the generalization, we take $\alpha_1 = 16R$, $\alpha_2 = \alpha_3 = 0$, $\alpha_4 = 1 - \gamma$, together with $c_1 = d_0 = 0$, $a_0 = c_0 = 16R$, and $a_2 = c_2 = 1$, where R and γ are adjustable parameters. With these simplifications, (8) becomes

$$(16R + \delta_y^4)(16R + \delta_x^4)u^{(n+1)} = [(16R + \delta_y^4)(16R + \delta_x^4) - 16R(\delta_x^4 + 2\delta_x^2\delta_y^2 + \delta_y^4) - (1 - \gamma)\delta_x^4\delta_y^4]u^{(n)}, \quad (9)$$

and (5) reduces to

$$(16R + \delta_y^4)u^{(n+\frac{1}{2})} = (-256R^2 + 16\gamma R\delta_x^4 + 32R\delta_x^2\delta_y^2)u^{(n)}$$
$$u^{(n+\frac{1}{2})} + (16R + \delta_x^4)u^{(n+1)} = (\gamma\delta_x^4)u^{(n)}. \quad (10)$$

Since $\alpha_2 = \alpha_3 = 0$, it can be seen that the sixth-order differences disappear in (7) and so the generalized method described by (9) or (10) converges to the solution of a difference approximation of the biharmonic equation which agrees with the standard difference form

$$(\delta_x^4 + 2\delta_x^2\delta_y^2 + \delta_y^4)u = 0,$$

except for a term involving eighth differences. This term will in general have a negligible effect on truncation whereas its inclusion can be used to accelerate the convergence of the alternating-direction method. The term, of course, disappears when $\gamma = 1$ and (9) becomes (4) with $R = \frac{1}{16r}$.

Convergence of the iterative procedure

The error $\epsilon_{i,j}^{(n)}$ is defined by

$$\epsilon_{i,j}^{(n)} = u_{i,j}^{(n)} - u_{i,j},$$

where $u_{i,j}$ is the solution of

$$\{16R(\delta_x^4 + 2\delta_x^2\delta_y^2 + \delta_y^4) + (1 - \gamma)\delta_x^4\delta_y^4\}u_{i,j} = 0. \quad (11)$$

The error growth is governed by equation (9) with u replaced by ϵ, together with homogeneous boundary conditions. If the error is expanded in the form

$$\epsilon_{i,j}^{(n)} = \rho_n \sin \pi p x_i \sin \pi q y_i \quad (p, q = 1, 2, \ldots, N - 1)$$

and substituted into (9) with u replaced by ϵ, it follows that

$$\lambda = \frac{\rho_{n+1}}{\rho_n} = \frac{R^2 - 2Rs_p^2s_q^2 + \gamma s_p^4s_q^4}{(R + s_p^4)(R + s_q^4)}, \quad (12)$$

where s_p and s_q are $\sin\frac{p\pi h}{2}$ and $\sin\frac{q\pi h}{2}$, respectively.

To facilitate the examination of the amplification factor (12), we introduce $\bar{\lambda}$, where

$$\bar{\lambda} = \frac{R^2 - 2Rs_p^2s_q^2 + \gamma s_p^4s_q^4}{R^2 + 2Rs_p^2s_q^2 + s_p^4s_q^4}. \quad (13)$$

Since $\qquad s_p^4 + s_q^4 \geqslant 2s_p^2s_q^2,$

it follows that $\qquad |\bar{\lambda}| \geqslant |\lambda|$

for all p and q. Put $z = s_p^2s_q^2$ and (13) becomes

$$\bar{\lambda}(R, \gamma, z) = \frac{R^2 - 2Rz + \gamma z^2}{R^2 + 2Rz + z^2}. \quad (14)$$

In what follows, the convergence of the process will always be based on $\bar{\lambda}$. Since $|\lambda| \leqslant |\bar{\lambda}|$, the actual convergence will generally be better than the figure quoted.

Optimum convergence factor

We now examine (14) with $R > 0$, $0 \leqslant z_1 \leqslant z \leqslant z_2$, and $\gamma \leqslant 1$, where $z_1 = \sin^4\frac{\pi}{2N}$ and $z_2 = \cos^4\frac{\pi}{2N}$. A typical graph of $\bar{\lambda}$ against z for given R and γ is illustrated in Fig. 1. This curve has a minimum at $z = \frac{2}{\gamma + 1}R$ where the value of $\bar{\lambda}$ is

$$-\frac{1 - \gamma}{3 + \gamma}(\equiv G_3) \quad (15c)$$

and $\bar{\lambda}$ tends to γ as z tends to infinity. The values of $\bar{\lambda}$ at the limits of z are

$$\frac{R^2 - 2Rz_1 + \gamma z_1^2}{R^2 + 2Rz_1 + z_2^2} (\equiv G_1) \qquad (15a)$$

and

$$\frac{R^2 - 2Rz_2 + \gamma z_2^2}{R^2 + 2Rz_2 + z_2^2} (\equiv G_2) \qquad (15b)$$

respectively. Depending on the values of R and γ, the minimum of the curve may lie inside or outside the permissible range of values of z. If the minimum lies outside, the maximum modulus value of z is either $|G_1|$ or $|G_2|$. If the minimum lies inside, the maximum modulus value is $|G_1|$, $|G_2|$, or $-G_3$.

In fact it can be shown after some manipulation that if γ_c is given by

$$\frac{32 \cos^2 \pi/N}{(1 + \cos^2 \pi/N)^2} = (\gamma_c + 3)^2(1 - \gamma_c), \qquad (16)$$

the convergence factors (maximum modulus values of $\bar{\lambda}$) are:

(1) $1 \geqslant \gamma \geqslant \gamma_c$

$|G_2|$ if $0 < R \leqslant R_c$

$|G_1|$ if $R_c < R$

where

$$R_c = \tfrac{1}{8}[\{(1 + \gamma)^2(z_1 + z_2)^2 + 32(1 + \gamma)z_1z_2\}^{1/2} - (1 - \gamma)(z_1 + z_2)]$$

(2) $\gamma_c > \gamma$

$|G_2|$ if $0 \leqslant R \leqslant R_1$

$|G_3|$ if $R_1 < R \leqslant R_2$

$|G_1|$ if $R_2 < R$,

where

$$R_1 = \frac{2 - (\gamma + 3)\{\frac{1}{2}(1 - \gamma)\}^{1/2}}{(\gamma + 1)} z_2$$

and

$$R_2 = \frac{2 + (\gamma + 3)\{\frac{1}{2}(1 - \gamma)\}^{1/2}}{(\gamma + 1)} z_1 \quad (R_2 > R_1).$$

In order to obtain the optimum convergence factor for a given value of N, it is necessary to find the values of γ and R which will minimize the maximum modulus values of $\bar{\lambda}$. In fact the minimum occurs where $\gamma = \gamma_c$ and $R = R_1 = R_2 = R_c$. This leads to an optimum convergence factor of $\frac{1 - \gamma_c}{3 + \gamma_c}$. Thus for any value of N, γ lies in the range

$$0 < \gamma_c \leqslant \gamma \leqslant 1$$

where γ_c is given by (16). The convergence factor is worst when $\gamma = 1$ and best when $\gamma = \gamma_c$.

The case of 100 internal nodes ($N = 10$) is illustrated in Table 1. As γ decreases from 1 (the C.D. value) to 0·24 (γ_c correct to two places), the convergence factor

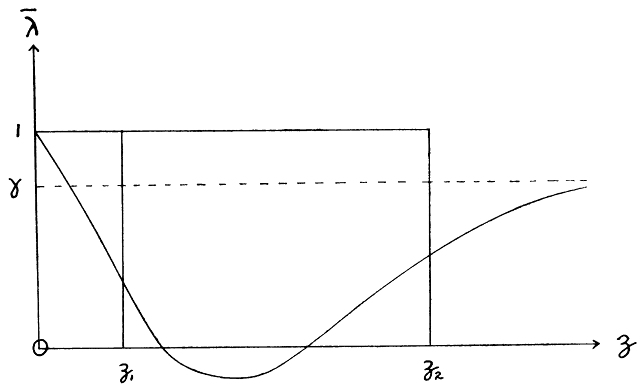

Fig. 1.

improves from 0·90 to 0·23, both values correct to two places. It should be noted, however, that a decrease in γ leads to a decrease in R and an increase in $(1 - \gamma)$, and so the eighth-order difference term in (11) becomes relatively more important. This will have little effect on the accuracy of the limiting solution of the iterative process, except possibly in circumstances when N is very large and γ has a value close to γ_c.

Final remarks

Conte and Dames, of course, did not advocate that R should be kept constant during iterations. In fact, they obtained a set of iteration parameters $R_k (k = 1, 2, \ldots n)$ which for $N = 10$, reduced the error by approximately 10^{-6} for 18 double sweeps over the grid.

In comparison it can be seen from Table 1 that the optimum convergence factor for the method outlined in the present paper is 0·235 when $N = 10$, which reduces the error by approximately 10^{-6} after 10 double sweeps, or by approximately 10^{-11} after 18 double sweeps. Alternatively, an error reduction of approximately 10^{-6} after 18 double sweeps can be obtained by choosing $\gamma = 0 \cdot 5$, which from Table 1 leads to a convergence

Table 1. ($N = 10$)

γ	$1 - \gamma$	R	CONVERGENCE FACTOR
1	0	0·02387	0·9045
0·875	0·125	0·01261	0·8190
0·75	0·25	0·007447	0·7187
0·50	0·50	0·003489	0·4874
0·25	0·75	0·001973	0·2440
0·2397	0·7603	0·001931	0·2347

factor of 0·49. Also there is no reason why the convergence of the present method should not be improved by making use of a variable parameter R. This has been employed successfully by several workers in the solution of Laplace's equation (see Varga, 1962) and, of course, by Conte and Dames in solving the biharmonic equation.

The original Douglas–Rachford method for solving equations of the Laplace type required a tridiagonal system of equations to be solved twice for each double sweep. The method described by (10) with the appropriate coefficients requires a quidiagonal system of equations to be solved twice for each double sweep. The method of solution of equations of type (10) is fully described by Conte and Dames.

Finally, it should be pointed out that the method proposed here for solving the biharmonic equation can be used for boundary conditions other than those described by (2). If the region departs from the rectangular, however, the method will require further justification (cf. Varga, 1962) and the convergence factors quoted will require modification.

Acknowledgement

Mr. G. Fairweather's share of the work was performed whilst in receipt of a Carnegie Scholarship.

References

CONTE, S. D., and DAMES R. T. (1958). "An alternating direction method for solving the biharmonic equation," *M.T.A.C.*, Vol. 12, pp. 198–205.

DOUGLAS, J., and RACHFORD, H. H. (1956). "On the numerical solution of heat conduction problems in two and three space variables," *Trans. Amer. Math. Soc.*, Vol. 82, pp. 421–439.

TEE, G. J. (1963). "A novel finite-difference approximation to the biharmonic operator," *The Computer Journal*, Vol. 6, pp. 177–192.

VARGA, R. S. (1962). *Matrix Iterative Analysis*, New Jersey, Prentice-Hall.

WHITE, R. N. (1963). "Optimum solution techniques for finite–difference equations," *Proc. A.S.C.E.*, Vol. 89, pp. 115–136.