Estimation of the line over-relaxation factor and convergence rates of an alternating direction line over-relaxation technique

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In this paper, an alternating direction line over-relaxation process is applied to the Dirichlet problem for the Laplace difference equation following the recent successful approach of Garabedian (1956). Estimates for the line over-relaxation factor and the convergence rate for both the 5-point and 9-point finite-difference equations are derived which agree closely with experimental results.

The technique first used by Garabedian (1956), and later by others, including Young (1962), Varga (1962), and Evans (1962a), for the estimation of over-relaxation factors for iterative methods such as S.O.R. and S.L.O.R. has shown itself to be extremely useful and more general in application than the rigorous theory developed by Young (1954). It is well known, for instance, that for the familiar 9-point finite-difference formula the matrix of coefficients does not possess Young's "Property A". Nevertheless, using the approach of Garabedian we are able to obtain a formula for the relaxation factor which agrees closely with experimental results. His technique may briefly be described as follows. He regards successive iterations of the iterative process as time steps to give an analogous hyperbolic difference equation, from which a good estimation of the over-relaxation factor is obtained by maximizing the decay of time-dependent terms in the solution of the equation. In the present paper a similar theory is applied to an alternating direction line over-relaxation process, previously described in Evans (1962b), and for which no adequate theory has been obtained. It is shown that good agreement with experimental results can be obtained. This theory is dependent on the assumption that the mesh size approaches zero.

The model problem of the Laplace partial differential equation over the unit square with prescribed boundary values is again chosen so that comparison may be made with earlier work on S.O.R. and S.L.O.R. methods. The usual procedure of replacing the partial differential equation by a system of finite-difference equations based on a discrete uniform mesh of size h reduces the problem to one of solving a large set of linear equations in which the coefficient matrix is sparse.

Subscripts i and j are used to refer to the column and row locations of a point on the grid, and if, as in Evans (1962a), we express the 5-point finite-difference equation at the point (i, j) in the form

$$\phi_{i,j}^{(n+1)} = \phi_{i,j}^{(n)} + (\beta/4)[\phi_{i-1,j}^{(n+1)} + \phi_{i,j-1}^{(n+1)} + \phi_{i,j+1}^{(n+1)} + \phi_{i+1,j}^{(n)} - 4\phi_{i,j}^{(n)}] \quad (1)$$

and group all such equations (1) for all the points along the column i, we obtain a group of $(N - 1)$ equations of the form

$$\phi_{i,j-1}^{(n+1)} - (4/\beta)\phi_{i,j}^{(n+1)} + \phi_{i,j+1}^{(n+1)} = -[\phi_{i-1,j}^{(n+1)} - 4(1 - 1/\beta)\phi_{i,j}^{(n)} + \phi_{i+1,j}^{(n)}] \quad (2)$$

$$(i, j = 1) \text{ to } (i, j = N - 1)$$

in which the coefficient matrix is of tridiagonal form. Such systems of simultaneous equations are efficiently solved by a method described by Evans and Forrington (1963) which has some advantages over, but less generality than the method of Cuthill and Varga (1959).

When we consider all such like systems of equations along the columns of the network we obtain the method of successive line over-relaxation, S.L.O.R., whereby, on any particular column, new values at pivotal points are computed simultaneously from the most recent values on the grid, and the iterative process moves successively from column $i = 1$ to $i = N - 1$. In the above equations, β is a suitably chosen constant termed the *line over-relaxation factor*, superscript n refers to the present (known) iterate on the grid, and we are in the process of determining the $(n + 1)$th iterate. At any stage of the whole iterative process, only one set of iterates, past or present, is needed for the calculation to proceed, a consideration which makes the method more adaptable for use on high-speed computers. The determination of β, which is used for speeding up the convergence rate of the iterative process, is important; the theoretical considerations concerning its choice, and the agreement with experimental results is given in Evans (1962a).

Let us now consider a variant to the above procedure which culminates in a method which is termed the *alternating direction successive line over-relaxation* process, and which consists of one iteration of the above process, whereby all the columns are processed in turn followed by a similar operation in which all the rows are processed in turn. The 5-point difference equation similar to (1) and suitable for row-wise treatment is

$$\phi_{i,j}^{(n+1)} = \phi_{i,j}^{(n)} + (\beta/4)[\phi_{i-1,j}^{(n+1)} + \phi_{i+1,j}^{(n+1)} + \phi_{i,j-1}^{(n+1)} + \phi_{i,j+1}^{(n)} - 4\phi_{i,j}^{(n)}]. \quad (3)$$

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The relevant equations for the two-step iterative method are thus:

$$\phi_{i,j-1}^{(2m+1)} - (4/\beta)\phi_{i,j}^{(2m+1)} + \phi_{i,j+1}^{(2m+1)} = - [\phi_{i-1,j}^{(2m+1)} - 4(1 - 1/\beta)\phi_{i,j}^{(2m)} + \phi_{i+1,j}^{(2m)}] \quad (4)$$

for $(i, j = 1)$ to $(i, j = N - 1)$ and each column $i = 1(1)N$, followed by

$$\phi_{i-1,j}^{(2m+2)} - (4/\beta)\phi_{i,j}^{(2m+2)} + \phi_{i+1,j}^{(2m+2)} = - [\phi_{i,j-1}^{(2m+2)} - 4(1 - 1/\beta)\phi_{i,j}^{(2m+1)} + \phi_{i,j+1}^{(2m+1)}] \quad (5)$$

for $(i = 1, j)$ to $(i = N - 1, j)$ and each row $j = 1(1)N$,

where m now denotes an iteration count in either direction of line iteration (row or column).

We see immediately that the new method enjoys all the advantages of the previous method and our immediate concern is now the choice of β to optimize the convergence rate and its comparison with previous methods such as S.O.R. and S.L.O.R. with known convergence rates.

We proceed as before and express β in the form

$$\beta = 4/3(1 + Ch) \quad (6)$$

where h is the mesh spacing and C a positive quantity, and on substitution in (1) we see that it approximates the hyperbolic partial differential equation

$$\psi_1(x, y, t_1) \equiv \phi_{xx} + \phi_{yy} - \phi_{xt_1} - 3C\phi_{t_1} = 0 \quad (7)$$

for small h. (See Evans (1962*a*).)

Similarly, when we substitute β into equation (3), we see that it approximates the hyperbolic equation

$$\psi_2(x, y, t_2) \equiv \phi_{xx} + \phi_{yy} - \phi_{yt_2} - 3C\phi_{t_2} = 0 \quad (8)$$

under similar discretization conditions, where in this case $t_2 = t_1 + h$. Now, when these two line iterations (row and column) are combined to form an alternating direction line iteration process as described in Section 2, the values $\psi_1(x, y, t_1)$ and $\psi_2(x, y, t_2)$ are the results given by a difference differential equation on a grid of size $t_2 - t_1 = h$.

If we now denote by $\psi(x, y, t)$, the partial differential equation analogue representing the new method, then on expanding by Taylor series, these two relationships must be valid:

$$\psi_1 = \psi + h\frac{\partial\psi}{\partial t} + O(h^2)$$

and

$$\psi_2 = \psi + 2h\frac{\partial\psi}{\partial t} + O(h^2),$$

from which we can determine ψ for small h by neglecting terms of $O(h)$.

Hence, the hyperbolic partial differential equation

$$\psi(x, y, t) \equiv \phi_{xx} + \phi_{yy} - 0{\cdot}5\phi_{xt} - 0{\cdot}5\phi_{yt} - 3C\phi_t = 0 \quad (9)$$

represents an approximation to the alternating direction line iteration for small h, where now each double iteration stage is considered as a time step of an unsteady problem, and index $2m$ refers to a time variable t, whilst index $(2m + 2)$ refers to a new time of $(t + 2h)$, where h is the mesh size.

Let us now introduce the new variable $s = t + x/4 + y/4$ so that $\phi(x, y, t) \equiv \phi(x, y, s - x/4 - y/4)$. On substitution into equation (9) this produces the equation

$$2\left(\phi_{xx} + \frac{1}{2}\phi_{xs} + \frac{1}{16}\phi_{ss} + \phi_{ys} + \frac{1}{2}\phi_{ys} + \frac{1}{16}\phi_{ss}\right) = \phi_{xs} + \frac{1}{4}\phi_{ss} + \phi_{ys} + \frac{1}{4}\phi_{ss} + 3C\phi_t$$

which, on summing terms, gives an equation of wave-propagation type

$$\phi_{xx} + \phi_{yy} - \frac{1}{8}\phi_{ss} - 3C\phi_t = 0. \quad (10)$$

Applying the method of separation of variables to (10), we obtain the result

$$\frac{X''}{X} + \frac{Y''}{Y} = \frac{1}{8}\frac{S''}{S} + 3C\frac{S'}{S} = -k_r^2 \quad (11)$$

where k_r^2 is a constant and $\phi(x, y, s) \equiv X(x)Y(y)S(s)$.

Finally, the representation

$$\phi(x, y, s) = \phi_0(x, y) + \sum_{m=1}^{\infty}(a_r e^{-p_r s} + b_r e^{-q_r s})\phi_r(x, y) \quad (12)$$

yields a solution $\phi(x, y, s)$ of (10), where ϕ_0 is the steady-state solution, a_m and b_m are Fourier coefficients,

$$p_r = 12C - 2\sqrt{2}(18C^2 - k_r^2)^{1/2},$$
$$q_r = 12C + 2\sqrt{2}(18C^2 - k_r^2)^{1/2} \quad (13)$$

and ϕ_r and k_r^2 are the eigenfunctions and eigenvalues of

$$\frac{\partial^2\phi_r}{\partial x^2} + \frac{\partial^2\phi_r}{\partial y^2} + k_r^2\phi_r = 0 \quad (14)$$

with homogeneous boundary conditions.

As before, the largest exponent of the time-dependent terms in (12) governs the decay process as t increases, i.e.,

$$p = \text{Re}\,[12C - 2\sqrt{2}(18C^2 - k_1^2)^{1/2}] \quad (15)$$

where k_1^2 is the smallest eigenvalue of (14).

We now choose the constant C so that the exponent p is a maximum, and hence the maximum rate of convergence is achieved for the iterative process. This occurs when $C = k_1/3\sqrt{2}$, which gives $p = 2\sqrt{2}k_1$ for the exponent governing the convergence rate. Yet again, as in earlier references, an underestimate of C is less damaging to the convergence rate than an overestimate.

We now use the simplest estimate for k_1, under Dirichlet boundary conditions as given by Polya and Szego (1951). On substitution into (1), the approximate formula for β becomes

$$\beta = \frac{4}{3(1 + 1\cdot003hA^{-1/2})} \tag{16}$$

for small h, where A is the area of the region.

The iterative process (4) and (5) described in Section 2 was programmed for the chosen problem with mesh sizes $h^{-1} = 15$, 20, 30 and 40 and run on the Sheffield University Mercury computer, and the optimal β sought experimentally. The results for the case $h^{-1} = 20$ are shown in Fig. 1, with the accompanying results for S.L.O.R. shown for comparison. The number of iterations required to give a specified accuracy from a given arbitrary initial solution is shown as a function of β, and the optimal β is clearly indicated in each case. These experimental results were compared with the results given theoretically by equation (16) and good agreement was obtained for small h. These results are given in Table 1 below. The divergence between experimental and theoretical results becomes significant as h increases in value, and supports the theoretical analysis that h must be small. However, the present divergence is appreciably greater than that found in the results for S.L.O.R. (Evans (1962a)). However, the alternating direction method consists of a double step process and in the approximation to the analogous hyperbolic equation the process is regarded as possessing time steps of $2h$. Hence we would now expect to find comparable agreement on accuracy with twice the value of h as that found previously for the S.L.O.R. method. This is found to be true on scrutiny of the results given here and elsewhere (Evans 1962a).

The application of the alternating direction line iteration method to other regions with assorted mesh sizes has been investigated experimentally and is given in Evans (1962b).

Table 1

h (MESHSIZE)	OPTIMAL β (THEORETICAL)	OPTIMAL β (EXPERIMENTAL)
0·06667	1·26	1·29
0·05	1·273	1·3
0·03333	1·29	1·305
0·025	1·3	1·307
0·02	1·31	1·31

By a similar approach, we can examine the alternating direction line iteration method when it is applied to the solution of 9-point finite-difference equations. This application is quite important because of the smaller truncation error of the 9-point formula. Consider the equations:

$$\begin{aligned}\phi_{1,j}^{(2m+1)} = \phi_{i,j}^{(2m)} &+ (\beta/20)[\phi_{i-1,j-1}^{(2m+1)} + 4\phi_{i-1,j}^{(2m+1)} \\ &+ \phi_{i-1,j+1}^{(2m+1)} + 4\phi_{i,j-1}^{(2m+1)} + 4\phi_{i,j+1}^{(2m+1)} \\ &+ \phi_{i+1,j-1}^{(2m)} + 4\phi_{i+1,j}^{(2m)} + \phi_{i+1,j+1}^{(2m)} - 20\phi_{i,j}^{(2m)}]\end{aligned} \tag{17}$$

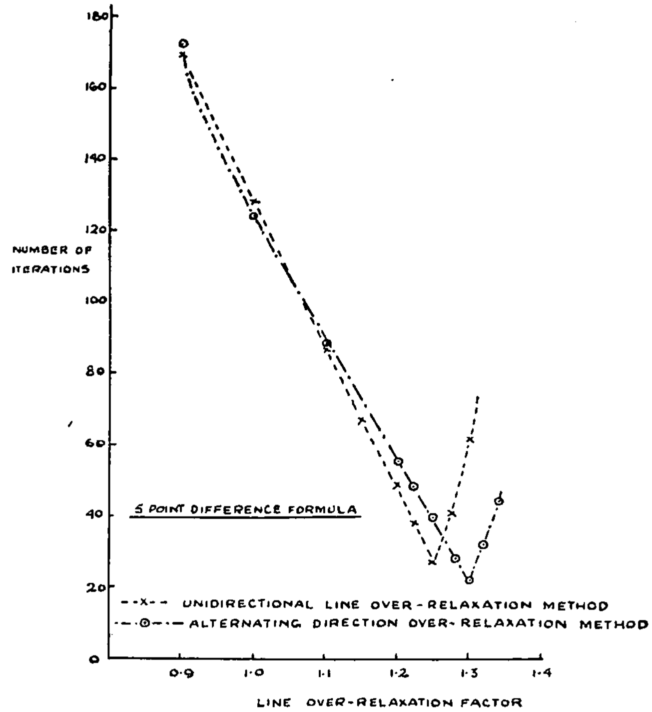

Fig. 1

for $(i, j = 1)$ to $(i, j = N - 1)$
and each column $i = 1(1)N - 1$,

and

$$\begin{aligned}\phi_{i,j}^{(2m+2)} = \phi_{i,j}^{(2m+1)} &+ (\beta/20)[\phi_{i-1,j-1}^{(2m+2)} + 4\phi_{i,j-1}^{(2m+2)} \\ &+ \phi_{i+1,j-1}^{(2m+2)} + 4\phi_{i-1,j}^{(2m+2)} + 4\phi_{i+1-j}^{(2m+2)} \\ &+ \phi_{i-1,j+1}^{(2m+1)} + 4\phi_{i,j+1}^{(2m+1)} + \phi_{i+1,j+1}^{(2m+1)} - 20\phi_{i,j}^{(2m+1)}]\end{aligned} \tag{18}$$

for $(i = 1, j)$ to $(i = N - 1, j)$
and each row $j = 1(1)N-1$.

The equations must be arranged similarly to (4) and (5) for the solution process. Further, it can be easily verified that when β is expressed as $10/7(1 + Ch)$ and applied to equations (17) and (18) in turn, the analysis leads to the hyperbolic partial differential equations

$$3\phi_{xx} + 3\phi_{yy} = 3\phi_{xt_1} + 7C\phi_{t_1} \tag{19}$$

and

$$3\phi_{xx} + 3\phi_{yy} = 3\phi_{yt_2} + 7C\phi_{t_2} \tag{20}$$

from which we can infer that, under similar assumptions as those given earlier, the hyperbolic equation

$$3\phi_{xx} + 3\phi_{yy} = 1\cdot5\phi_{xt} + 1\cdot5\phi_{yt} + 7C\phi_t \tag{21}$$

represents a finite-difference analogue of the alternating direction line over-relaxation method when applied to 9-point finite-difference equations.

By a similar analysis we can deduce that the most rapid convergence occurs when $C = 3\sqrt{2k_1}/14$ and the value of the exponent p governing the convergence rate is $2\sqrt{2k_1}$.

Finally, the value of β for the 9-point finite-difference formula is

$$\beta = \frac{10}{7(1 + 1 \cdot 29hA^{-1/2})} \tag{22}$$

for small h.

The iteration scheme (17) and (18) for $h^{-1} = 20$ for the chosen model problem is shown in Fig. 2 with the corresponding result for S.L.O.R. given for comparison; good agreement with the theoretical result given by equation (22) was obtained for the optimal β.

That the rates of convergence for both the 5-point and 9-point formula are equal is given both by the theoretical analysis $p = 2\sqrt{2}\, k_1$, and the experimental results shown in Figs. 1 and 2. This rate of convergence compares with the result $p = 2k_1$ for the S.L.O.R. method (Figs. 1 and 2) and $p = \sqrt{2}k_1$ for the S.O.R. method (Garabedian 1956). Hence, the alternating direction line iteration method at the optimal β is a factor of $\sqrt{2}$ faster than the S.L.O.R. method and a factor of 2 faster than the S.O.R. method from this theoretical analysis. Good agreement with these results is obtained by comparing Figs. 1 and 2 with Fig. 2 of Evans (1962a).

The work done per iteration must now be compared. Since this has not been increased in going from point to line methods, the methods discussed are directly comparable and the gains from these more recent methods (line and alternating direction line iteration) are attainable.

The application of this alternating direction technique to the 2-line and 3-line iteration methods of Varga (1962) and Parter (1961) is fairly obvious, and can bring about further increases in the convergence rates of these iterative processes when used for the numerical solution of self-adjoint elliptic partial differential equations in two dimensions.

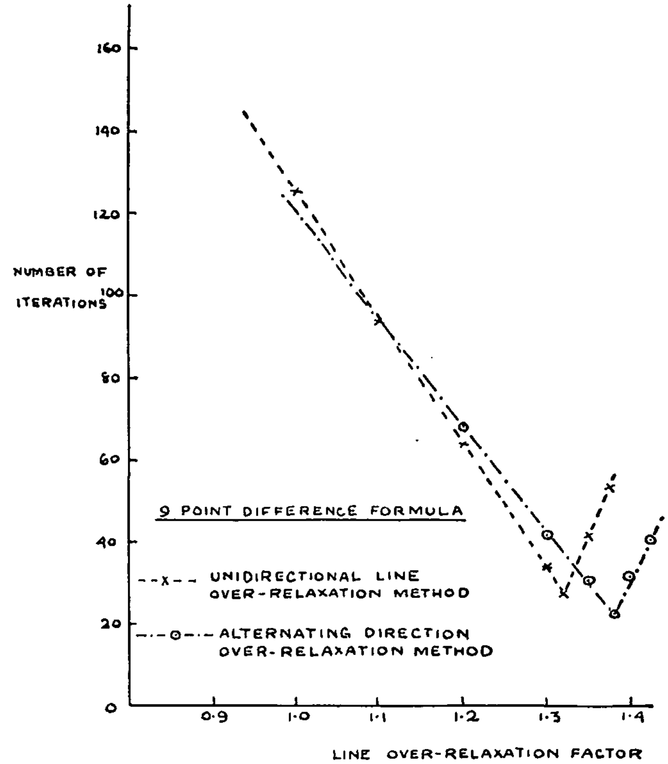

Fig. 2

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