Obtaining solutions of the Navier-Stokes equation by relaxation processes

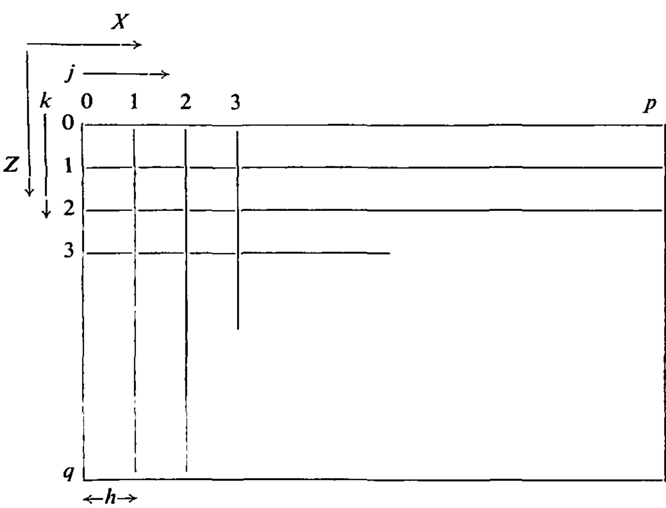
By John A. T. Bye*

The results of a series of relaxations of the Navier–Stokes equation by a special line iteration process are presented. The convergence properties are similar to those of the Biharmonic equation at low Reynolds Numbers, but at higher Reynolds Numbers the non-linearity of the equation severely reduces the maximum convergence rate.

1. Introduction

In recent years there has been much theoretical discussion in the literature of relaxation techniques for solving the Biharmonic equation (Frankel, 1950; Windsor, 1957; Conte and Dames, 1958; Parter, 1959). In this note we construct non-linear finite-difference approximations to the Navier–Stokes equation with specified boundary conditions over a rectangle, and we examine the convergence properties of a special line iterative procedure which was applied for solving these equations. The computation was performed on the Ferranti Mercury computer at London University in 1960–61.

A series of solutions at different Reynolds Numbers was calculated, for finite-difference networks of $p \times 6$ internal nodes (where p was large, greater than 20). Satisfactory convergence rates were observed at low Reynolds Numbers, but at higher Reynolds Numbers, due to the increasing non-linearity of the equations, it was found that the maximum possible convergence rate rapidly decreased.



$j \rightarrow 1$ to $p - 1$, $k \rightarrow 1$ to $q - 1$, and h is the mesh interval

Fig. 1.

2. The problem

The purpose of the investigation was to calculate the circulations induced by a steady uniform stress acting on the surface of a liquid in a closed basin of vertical sides and uniform depth (Bye, 1962). The problem consisted of solving the two-dimensional Navier–Stokes equation, for circulation in a vertical plane parallel to the surface stress (F_s), at a series of values of Reynolds Numbers (Re) between 0 and 400.

Formulated mathematically, the problem is a solution of the non-linear partial differential equation:

$$\nabla^4\phi = 4Re\left(-\frac{\partial\phi}{\partial Z}\frac{\partial}{\partial X} + \frac{\partial\phi}{\partial X}\frac{\partial}{\partial Z}\right)\nabla^2\phi \qquad (1)$$

in a rectangular region, $X \rightarrow 0$ to A, $Z \rightarrow 0$ to 1 ($A \gg 1$), with the boundary conditions

$$\left.\begin{array}{ll} \phi = 0, \dfrac{\partial\phi}{\partial X} = 0, X = 0, A & 0 < Z < 1 \\ \phi = 0, \dfrac{\partial\phi}{\partial Z} = 0, Z = 1, & 0 < X < A \\ \phi = 0, \dfrac{\partial^2\phi}{\partial Z^2} = 1, Z = 0, & 0 < X < A \end{array}\right\} \qquad (2)$$

where ϕ is a non-dimensional stream-function and

$$Re = \frac{F_s D^2}{4\rho\nu^2}.$$

(D is the depth of the basin, ρ is the density, and ν is the dynamic viscosity of the liquid.)

This paper describes the method used to obtain most of the solutions between Reynolds Numbers 0–200.

3. The finite-difference equations

Firstly the equation was approximated by a set of second-order finite-difference equations which were solved for each interior point (j, k) of the rectangular network shown in **Fig. 1.**

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The finite-difference stencil at network points, except for those adjacent to the boundary, is

$$\frac{1}{h^4}\begin{bmatrix} & & 1 & & \\ & 2 & -8 & 2 & \\ 1 & -8 & 20 & -8 & 1 \\ & 2 & -8 & 2 & \\ & & 1 & & \end{bmatrix} \phi(j,k) - \frac{Re}{h^4}$$

$$\left[\begin{matrix} & & & & \\ & -1 & 1 & 1 & \\ -1 & 4 & 0 & -4 & 1 \\ & -1 & -1 & 1 & \\ & & & & \end{matrix} + \begin{matrix} & -1 & \\ -1 & 4 & -1 \\ -1 & 0 & 1 \\ 1 & -4 & 1 \\ & 1 & \end{matrix}\right] \phi(j,k)$$

$$+ \mathrm{O}(h^2) = 0. \qquad (3)$$

In the second term of the stencil, the coefficients in circles represent two-term difference operators which multiply the remaining coefficients.

In the columns $j = 1$ and $p - 1$, and the rows $k = 1$ and $q - 1$, this stencil involves external nodes. These nodes were eliminated before iteration commenced by combining with the finite-difference boundary conditions. For example, for points in the top row $k = 1$, except the points $(1, 1)$ and $(p - 1, 1)$ the boundary conditions modify the stencil to

$$\frac{1}{h^4}\begin{bmatrix} 1 & -8 & 19 & -8 & 1 \\ & 2 & -8 & 2 & \\ & & 1 & & \end{bmatrix} \phi(j,1) - \frac{Re}{h^4}$$

$$\left[\begin{matrix} -1 & 4 & 0 & -4 & 1 \\ & -1 & -1 & 1 & \\ & & & & \end{matrix} + \begin{matrix} -1 & 1 & 1 \\ 1 & -4 & 1 \\ & 1 & \end{matrix}\right] \phi(j,1)$$

$$+ \frac{1}{h^2} + \frac{Re}{h^2}[-1 \;\; 0 \;\; 1]\; \phi(j,1) + \mathrm{O}(h^2) = 0. \qquad (4)$$

Here, the nodes in the top row of the stencil (3) have been eliminated by the boundary condition $\dfrac{\partial^2\phi}{\partial Z^2} = 1$, and nodes in the second row set identically zero by the boundary condition, $\phi = 0$.

In the solutions to be described, the truncation term $\mathrm{O}(h^2)$ was ignored. A trial solution showed that little advantage was gained, but much extra labour was involved, in applying a second-order difference correction.

4. The numerical method of solution

The vertical line (or column) iteration method used to find the solution was extremely simple. Before iteration was started a single line biharmonic iteration matrix A of dimensions $(q - 1) \times (q - 1)$ and defined by the matrix equation (5) was formed and inverted.

$$A = \begin{bmatrix} 19 & -8 & 1 & & & & & \\ -8 & 20 & -8 & 1 & & & & \\ 1 & -8 & 20 & -8 & 1 & & & \\ & & \cdot & & \cdot & & & \\ & & & & 1 & -8 & 20 & -8 & 1 \\ & & & & & 1 & -8 & 20 & -8 \\ & & & & & & 1 & -8 & 21 \end{bmatrix}. \qquad (5)$$

The network was then iterated by columns. A complete cycle of the iterative process consisted of scanning over all columns in turn in the order

$$j = 1, 2, \ldots, \ldots, p - 2, p - 1, p - 2, \ldots, 2, 1.$$

In each iteration ϕ was relaxed simultaneously at all nodes within the column, and the new ordinates $\phi^{(2n+i+1)}(j, k)$ were computed from the matrix equation (6).

$$\begin{bmatrix} \phi^{(2n+i+1)}(j,1) \\ \phi^{(2n+i+1)}(j,2) \\ \cdot \\ \cdot \\ \cdot \\ \phi^{(2n+i+1)}(j,q-1) \end{bmatrix} = \begin{bmatrix} \phi^{(2n+i)}(j,1) \\ \phi^{(2n+i)}(j,2) \\ \cdot \\ \cdot \\ \cdot \\ \phi^{(2n+i)}(j,q-1) \end{bmatrix} - \omega A^{-1} \begin{bmatrix} R_\phi^{(2n+i)}(j,1) \\ R_\phi^{(2n+i)}(j,2) \\ \cdot \\ \cdot \\ \cdot \\ R_\phi^{(2n+i)}(j,q-1) \end{bmatrix}. \qquad (6)$$

Here the residuals $R_\phi(j, k)$, are defined as $h^4 \times$ the value of the L.H.S. of the stencil equation (appropriately modified in the cases $j = 1$ and $p - 1$), and the vector of residuals was computed from the latest available estimates of ϕ, immediately before relaxing the current column. ω is the relaxation parameter, A^{-1} is the inverse of the iteration matrix, n is the number of the current double scan, and $i = 0$ for scanning by columns from $j = 1$ to $p - 1$ and $i = 1$ for scanning by columns from $j = p - 1$ to 1.

The solution started from arbitrary initial values $\phi^{(0)}(j, k)$.

After each scanning of the network the relaxation parameter (ω), the sum of the squares of the residuals,

$$G^{(n)} = \sum_{i=0}^{1} \sum_{j=1}^{p-1} \sum_{k=1}^{q-1} R_\phi^{2(2n+i)}(j,k)$$

and the quantity $\beta^{(n)}$ defined as

$$\beta^{(n)} = 1 - \left[\frac{G^{(n)}}{G^{(n-1)}}\right]^{1/4}$$

were recorded. It is readily seen that $\beta^{(n)}$ is a convenient measure of the convergence rate of the iterative procedure.

The relaxation parameter (ω) could be adjusted between scannings of the network by setting a hand-switch on the computer, and the convergence was studied by plotting $\beta^{(n)}$ against ω. Normally ω was gradually increased until its optimum value was found.

5. The convergence of the solution

(a) *The optimum relaxation parameter*

Two values of the parameter ω are particularly significant: the maximum parameter, above which the iterative process became unstable, and the optimum parameter $\omega^X(Re)$, i.e. the value of ω at which the iterative procedure converges the most rapidly, for a fixed value of *Re*. For Reynolds Numbers > 40, the optimum was only just less than the maximum parameter; the transition from convergence to severe instability being very sharp and occurring within a change in ω of less than $0{\cdot}02$. Both parameters decreased as the Reynolds Number increased, so that for Reynolds Numbers > 80 the optimum ω was less than 1. The observed values are recorded in **Table 1.**

In practice, because of the small difference between the optimum and the maximum parameter, the solutions were obtained usually only with parameters up to slightly below the optimum.

(b) *The convergence rate*

Below the optimum parameter, the convergence rate ($\beta^{(n)}$) depended on the state of the convergence. In the early part of the convergence, high rates were observed at all parameters. If, however, the relaxation parameter was held constant for sufficient scannings of the network, the convergence rate at that parameter tended to a limiting value ($\bar{\beta}$).

From estimates of $\bar{\beta}$ at several relaxation parameters, a graph of $\bar{\beta}$ against ω for each solution was drawn. A noteworthy feature of the graphs was that there was no observable variation in the convergence rate ($\bar{\beta}$) with Reynolds Number.

6. Comparison with theoretical results

Parter (1959) obtained an estimate of the effective decay factor (λ_L) for the two-line Liebmann method for the biharmonic equation with ϕ and $\dfrac{\partial\phi}{\partial n}$ specified everywhere on the boundaries.

$$\lambda_L \leqslant 1 - \pi^4(p^{-2} + q^{-2})^2 \qquad (7)$$

for which, applying the theoretical definition of the convergence rate,

$$\rho = -\ln\lambda$$

the unextrapolated convergence rate (ρ_L) is:

$$\rho_L \leqslant \pi^4(p^{-2} + q^{-2})^2. \qquad (8)$$

Since Parter's equations are consistently ordered within Block-property *A*, the eigenvalues λ_E of the two-line S.O.R. method for his problem obey Young's Formula

$$(\lambda_E + \omega - 1)^2 = \omega^2\lambda_L\lambda_E \qquad (9)$$

(cf. Parter, 1959).

Taking logarithms in equation (9), and substituting $\rho_L = -\ln\lambda_L$ where ρ_L is the unextrapolated convergence rate, and $\rho = -\ln\lambda_E$ where ρ is the S.O.R. convergence rate, we obtain

$$-\rho - \rho_L = 2\ln\left(\frac{e^{-\rho} + \omega - 1}{\omega}\right)$$

$$= 2\ln\left(\frac{\omega - \rho + O(\rho^2)}{\omega}\right)$$

$$= \frac{-2\rho}{\omega} + O(\rho^2)$$

therefore $$\rho = \frac{\rho_L'\omega/2}{1-\omega/2}$$

where $$\rho_L' = \rho_L + O(\rho^2).$$

Therefore if $\rho_L \to 0$ we have,

$$\rho \approx \frac{\rho_L\omega/2}{1-\omega/2} \qquad (10)$$

unless ω varies so that $1 - \omega/2 = O(\rho)$ in which case it follows from (10) that $\rho_L = O(\rho^2)$.

Now in all cases we find that the convergence rate ($\bar{\beta}$) of our solutions for $\omega \leqslant \omega^x(Re)$ could be fitted to the empirical formula:

$$\bar{\beta} = (0{\cdot}021 \pm 0{\cdot}001)\frac{\omega}{1 - (0{\cdot}50 \pm 0{\cdot}01)\omega} \qquad (11)$$

$$0 < \omega \leqslant \omega^x(Re).$$

This equation is of the same form as equation (10) despite the fact that our finite-difference equations (even in the linearized form at zero Reynolds Number) do not possess Block-property *A*.

Table 1

The maximum and optimum relaxation parameters and the optimum convergence rate of the solutions

REYNOLDS NUMBER	MAXIMUM RELAXATION PARAMETER	OPTIMUM RELAXATION PARAMETER	OPTIMUM CONVERGENCE RATE	p
0	$>1{\cdot}62$	$>1{\cdot}62$	$>0{\cdot}174$	56
20	$>1{\cdot}62$	$>1{\cdot}62$	$>0{\cdot}174$	56
40	$1{\cdot}45$	$1{\cdot}43$	$0{\cdot}115$	56
60	$1{\cdot}17$	$1{\cdot}16$	$0{\cdot}065$	109
100	$0{\cdot}94$	$0{\cdot}93$	$0{\cdot}038$	109
150	$>0{\cdot}4$	$>0{\cdot}4$	$>0{\cdot}01$	24
200	$0{\cdot}50$	$0{\cdot}50$	$0{\cdot}017$	24

Further, if we obtain the unextrapolated convergence rate ($\bar{\beta}_L$), by substituting $\omega = 1$ in equation (11),

$$\bar{\beta}_L = 0 \cdot 042 \pm 0 \cdot 002$$

and compare it with ρ_L (equation (8)), which for our network parameters, $p = 24 - 109$ and $q = 6$, varies between,

$$\rho_L \leqslant 0 \cdot 085 \text{ and } \rho_L \leqslant 0 \cdot 076,$$

it is clear that the convergence rate of our special line iterative method is about half Parter's estimated rate for the two-line Liebmann method for the Biharmonic equation under his boundary conditions.

Equation (11), however, unlike Young's Formula cannot be used to predict the optimum relaxation parameter. It is found that the optimum and maximum relaxation parameters of our convergences are determined by the non-linearity of the equations. In certain regions of the network, as the Reynolds Number increases, the coefficients of the side nodes of the stencil equation (3) become progressively greater than the central coefficient. This causes instability of the single line process at lower and lower relaxation parameters (Bye, 1962).

7. Conclusion

The convergence rate ($\bar{\beta}$) of this set of solutions of the Navier–Stokes equation by a special line iteration process varied with ω in approximately the same manner as the convergence rate of S.O.R. for consistently ordered matrices, provided that $\bar{\beta} < 2 - \omega$. The maximum relaxation parameter ω^x (Re), however, decreased steadily as the Reynolds Number increased. By Reynolds Number 200 the efficiency of the single line process had been seriously impaired.

It appears that if solutions are to be obtained effectively at higher Reynolds Numbers a more elaborate iteration scheme is required. This should not depend on the constant element iteration matrix (5), but take into account the local changes which occur in the coefficients of the nodes of the complete finite-difference stencil (3) as the convergence proceeds.

The development of such methods for the Navier–Stokes and other equations, although almost certainly demanding on computer time, may well make possible the numerical solution of important non-linear problems.

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Correspondence

To the Editor,
The Computer Journal.

"Irascible Genius"

Dear Sir,

Mr. Ord-Smith, in briefly reviewing Miss Moseley's book upon Charles Babbage, might have called attention to two astonishing misstatements regarding that mathematical genius the late Dr. A. M. Turing, F.R.S. On page 258 she says:

> Had he [Babbage] come back within seventy years, in 1936, he would have found another Englishman of genius, Alan Turing, wresting the touch from him and passing it on to others.

The reference is presumably to a paper entitled "On Computable Numbers" which Dr. Turing submitted to the London Mathematical Society on 28 May 1936, in which he discussed in terms of pure mathematics the computational limitations of such a machine as an electronic binary scale development of Babbage's analytical engine which had been discussed at the Institute of Actuaries four months earlier, namely on 27 January 1936.

Miss Moseley is not even consistent; in her Prologue, on page 17, she quotes the view that "Charles Babbage's ideas had begun to be properly appreciated only after the Second World War;" on the same page she refers to Turing as "the originator of the pilot ACE." Team work on the Pilot ACE commenced, to be pedantically precise, at 10.30 a.m. on Friday 15 January 1943. Turing did not join the team until the autumn of 1945.

Yours truly,
WILLIAM PHILLIPS.

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Temple, London, E.C.4.
4 February 1965.