The solution of linear differential equations in Chebyshev series

By R. E. Scraton*

Any linear differential equation can be transformed into an infinite set of simultaneous equations in the Chebyshev coefficients of its solution. In suitable cases these equations can be solved by iterative procedures, which are particularly suitable for automatic computation. Similar iterative methods can also be used for certain eigenvalue problems.

The numerical solution of the linear differential equation

$$y^{(n)} + \sum_{i=0}^{n-1} P_i(x)y^{(i)} = F(x) \tag{1}$$

in the form of a Chebyshev series

$$y = \tfrac{1}{2}a_0 + a_1T_1(x) + a_2T_2(x) + a_3T_3(x) + \ldots = \sum_{r=0}^{\infty}{}' \; a_rT_r(x)$$

has been the subject of a number of papers in recent years. The so-called 'direct' method of determining the coefficients a_r was first proposed by Clenshaw (1957), and has been subsequently discussed by Fox (1962). This method is only practicable when the functions $P_i(x)$ are polynomials of small degree. A later paper by Clenshaw and Norton (1963) gave an iterative procedure based on Picard's method and the principle of collocation; this is applicable to a wider class of differential equations, but is usually considerably more lengthy than the direct method.

The purpose of the present paper is to consider the extension of Clenshaw's original method to the case where the $P_i(x)$ are general functions with known Chebyshev expansions. It will be assumed throughout that the independent variable has been suitably transformed so that the solution is required in the range $-1 \leqslant x \leqslant 1$. It should be noted that in writing the differential equation in the form (1) and assuming that the $P_i(x)$ have Chebyshev expansions, it has been implicitly assumed that the differential equation has no singularities in the range $-1 \leqslant x \leqslant 1$.

First-order equations

The first-order linear equation may be put in the form

$$y' + P(x)y = F(x)$$

where $$P(x) = \sum_{r=0}^{\infty}{}' \; p_rT_r(x)$$

and $$F(x) = \sum_{r=0}^{\infty}{}' \; f_rT_r(x).$$

It will be assumed for the sake of definiteness that the initial condition is specified at $x = 1$, i.e. $y(1) = \eta$. The modifications to the equations below when the initial condition is specified at some other point should be reasonably obvious.

If in the usual way the Chebyshev coefficients of y are denoted by a_r, and those of y' by a'_r, it is easily seen that

$$2a'_r + \sum_{s=0}^{\infty}{}' \; (p_{r+s} + p_{|r-s|})a_s = 2f_r, \quad r \geqslant 0.$$

The a'_r term can be eliminated in the customary manner by making use of the relation

$$a'_{r-1} - a'_{r+1} = 2ra_r,$$

to yield

$$4ra_r + \sum_{s=0}^{\infty}{}' \; (\pi_{r+s} + \pi_{r-s})a_s = 2\phi_r, \quad r \geqslant 1, \tag{2}$$

where $\pi_r = p_{r-1} - p_{r+1}$, $\pi_0 = 0$, $\pi_{-r} = -\pi_r$

and $\phi_r = f_{r-1} - f_{r+1}$.

The initial condition is given by

$$\sum_{s=0}^{\infty}{}' \; a_s = \eta$$

and if this is used to eliminate a_0 equation (2) becomes

$$\sum_{s=1}^{\infty} (\pi_{r+s} - 2\pi_r + \pi_{r-s} + 4r\delta_{rs})a_s = 2(\phi_r - \eta\pi_r), \quad r \geqslant 1. \tag{3}$$

Now (3) is an infinite set of equations in the infinite set of unknowns $a_1, a_2, a_3, \ldots$ It is always possible to get approximate values for these unknowns by assuming that a_s is negligible for $s > n$, and solving the first n of the set of equations (3) for $a_1, a_2, \ldots a_n$. In fact Fox (1962) derived sets of equations equivalent to (3) for a few simple differential equations, and obtained solutions by this method. If, however, $P(x)$ is not too large—if, say, $P(x)$ is of order 1—the terms π_{r+s}, π_r, π_{r-s} in (3) will be small compared with the term $4r$. In other words, the set of equations will have a "strong diagonal." This suggests that an iterative procedure of the Gauss-Seidel type might be appropriate. Such a process would be defined as follows:

$$(4r + \pi_{2r} - 2\pi_r)a_r^{(k)} = 2(\phi_r - \eta\pi_r)$$

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Table 1

Steps in the iterative solution of $y' - \frac{1}{3}e^x y = 0$, $y(1) = 1$

k	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
1	0·3	0·1						
2	0·21	0·09	0·02					
3	0·248	0·083	0·023	0·006				
4	0·2477	0·0818	0·0220	0·0055	0·0013	0·0003		
5	0·24828	0·08194	0·02203	0·00540	0·00125	0·00028	0·00006	0·00001
6	0·24824	0·08194	0·02204					

$$- \sum_{s=1}^{r-1} (\pi_{r+s} - 2\pi_r + \pi_{r-s})a_s^{(k)}$$

$$- \sum_{s=r+1}^{\infty} (\pi_{r+s} - 2\pi_r - \pi_{s-r})a_s^{(k-1)} \quad (4)$$

where $a_r^{(k)}$ denotes the kth approximation to a_r. The initial approximations $a_r^{(0)}$ would normally be taken as zero. It is not necessary to decide beforehand how many of the coefficients a_r will be significant; at each stage in the iteration the a_rs are calculated successively, and those which are negligible to the number of decimal places being retained are ignored. The whole process reduces to the repetitive use of equation (4), and as such it is very simple to program for automatic computation.

As an example, **Table 1** shows the steps in the solution of the differential equation

$$y' - \tfrac{1}{3}e^x y = 0, \qquad y(1) = 1. \quad (5)$$

The final solution is

$$y = 0{\cdot}64078 + 0{\cdot}24824\, T_1(x) + 0{\cdot}08194\, T_2(x) + 0{\cdot}02204\, T_3(x) + 0{\cdot}00540\, T_4(x) + 0{\cdot}00125\, T_5(x) + 0{\cdot}00028\, T_6(x) + 0{\cdot}00006\, T_7(x) + 0{\cdot}00001\, T_8(x) + \ldots \quad (6)$$

The correct solution of (5) is $y = \exp[\frac{1}{3}(e^x - e)]$, so that $y(0) = 0{\cdot}563966$ and $y(-1) = 0{\cdot}456819$; the corresponding values of y given by the series (6) are 0·56397 and 0·45682 respectively.

Simultaneous differential equations

For the set of simultaneous differential equations

$$y'_m + \sum_{n=1}^{N} P_{mn}(x)y_n = F_m(x), \quad m = 1, 2, 3, \ldots N,$$

$$y_m(1) = \eta_m,$$

the equivalent of the set of equations (3) is

$$\sum_{n=1}^{N} \sum_{s=1}^{\infty} (\pi_{r+s}^{mn} - 2\pi_r^{mn} + \pi_{r-s}^{mn} + 4r\delta_{rs}\delta_{mn})a_s^n = 2(\phi_r^m - \sum_{n=1}^{N} \pi_r^{mn}\eta_n) \quad (7)$$

where $\qquad y_m = \sum_{r=0}^{\infty}{}' \, a_r^m T_r(x)$

and π_r^{mn}, ϕ_r^m are related to $P_{mn}(x)$, $F_m(x)$ in the same way that π_r, ϕ_r are related to $P(x)$, $F(x)$. This equation will, of course, need some changes if any of the initial conditions are specified at points other than $x = 1$, but the basic approach remains the same. The solution of the set of equations (7) may be obtained by an iterative procedure, as before.

Furthermore, since a differential equation of any order can be rewritten as a set of first-order simultaneous differential equations, the general linear equation (1) can be dealt with by a simple extension of the method described above for first-order equations.

The second-order equation $y'' + P(x)y = F(x)$

Equations of the form

$$y'' + P(x)\, y = F(x)$$

occur sufficiently often to merit special consideration. With the same notation as before

$$2a''_r + \sum_{s=0}^{\infty}{}' \, (p_{r+s} + p_{|r-s|})a_s = 2f_r, \quad r \geqslant 0,$$

and hence

$$4ra'_r + \sum_{s=0}^{\infty}{}' \, (\pi_{r+s} + \pi_{r-s})a_s = 2\phi_r, \quad r \geqslant 1.$$

After elimination of a'_r, this reduces to

$$8ra_r + \sum_{s=0}^{\infty}{}' \left[\frac{1}{r-1}(\pi_{r+s-1} + \pi_{r-s-1}) - \frac{1}{r+1}(\pi_{r+s+1} + \pi_{r-s+1})\right]a_s = 2\left[\frac{\phi_{r-1}}{r-1} - \frac{\phi_{r+1}}{r+1}\right], \quad r \geqslant 2. \quad (8)$$

The initial or boundary conditions attached to a particular problem give two more equations in the coefficients a_r; these may be used to eliminate a_0 and a_1 from (8), so as to leave a set of equations for the unknowns $a_2, a_3, a_4, \ldots$ For example, the initial-value problem

Table 2

Steps in the iterative solution of $y'' - (x^6 + 3x^2)y = 0,\ y(1) = y(-1) = 1$

k	a_2	a_4	a_6	a_8	a_{10}	a_{12}
1	0·114	0·030	0·002			
2	0·108870	0·030180	0·001758	0·000263	0·000014	0·000002
3	0·108828	0·030184	0·001757			

in which $y(1) = \eta$, $y'(1) = \eta'$ has the additional equations

$$\sum_{s=0}^{\infty}{}' a_s = \eta$$

and

$$\sum_{s=1}^{\infty} s^2 a_s = \eta'.$$

Equation (8) therefore becomes

$$\sum_{s=2}^{\infty}\left\{\frac{1}{r-1}[\pi_{r+s-1} - s^2\pi_r + 2(s^2-1)\pi_{r-1} - s^2\pi_{r-2} + \pi_{r-s-1}] - \frac{1}{r+1}[\pi_{r+s+1} - s^2\pi_{r+2} + 2(s^2-1)\pi_{r+1} - s^2\pi_r + \pi_{r-s+1}] + 8r\delta_{rs}\right\}a_s$$
$$= \frac{1}{r-1}[\phi_{r-1} - 2(\eta-\eta')\pi_{r-1} - \eta'(\pi_r + \pi_{r-2})] - \frac{1}{r+1}[\phi_{r+1} - 2(\eta-\eta')\pi_{r+1} - \eta'(\pi_{r+2} + \pi_r)]. \quad (9)$$

On the other hand, for the boundary-value problem in which $y(1) = \eta_1$ and $y(-1) = \eta_{-1}$, the additional equations are

$$a_0 + 2a_2 + 2a_4 + 2a_6 + \ldots = \mu$$

and

$$a_1 + a_3 + a_5 + a_7 + \ldots = \nu$$

where $\mu = \eta_1 + \eta_{-1}$ and $\nu = \frac{1}{2}(\eta_1 - \eta_{-1})$. In this case equation (8) can be put in the form

$$\sum_{s=2}^{\infty}(U_{r-1,s} - U_{r+1,s} + 8r\delta_{rs})a_s = \frac{1}{r-1}[2\phi_{r-1} - \mu\pi_{r-1} - \nu(\pi_r + \pi_{r-2})] - \frac{1}{r+1}[2\phi_{r+1} - \mu\pi_{r+1} - \nu(\pi_{r+2} + \pi_r)] \quad (10)$$

where $U_{r,s} = \frac{1}{r}(\pi_{r+s} - 2\pi_r + \pi_{r-s})$ if s is even

and $U_{r,s} = \frac{1}{r}(\pi_{r+s} - \pi_{r+1} - \pi_{r-1} + \pi_{r-s})$ if s is odd.

A similar procedure may be adopted in the more general case where the boundary conditions involve linear combinations of y and y'.

The sets of equations (9) and (10) may again be solved by an iterative Gauss–Seidel procedure. It should be noted, however, that the coefficient of a_s in (9) contains terms of order s^2. This means that the set of equations has large terms away from the diagonal, and consequently the efficiency of the iterative process will be diminished; indeed, in some cases the process may not converge at all. In these circumstances, it may be preferable to discard the iterative method and use an elimination procedure to solve the first $(n-1)$ of equations (9) for the unknowns $a_2, a_3, a_4, \ldots a_n$. A similar situation arises in boundary-value problems when the boundary conditions involve y' as well as y. When, however, the boundary conditions involve y only, the s^2 terms do not occur; it will be seen that the set of equations (10) has a very strong diagonal, and the iterative process is therefore rapidly convergent. (Strictly speaking, of course, the convergence of the Gauss–Seidel process depends on the magnitude of the latent roots of an associated matrix, rather than on the strength of the diagonal; but a strong diagonal is usually a fair indication that the process will converge quickly.)

As an example, the equation

$$y'' - (x^6 + 3x^2)y = 0, \quad y(1) = y(-1) = 1, \quad (11)$$

will be considered. The solution is clearly an even function of x, so that only $a_2, a_4, a_6, \ldots$ need be found. The steps in the solution are set out in **Table 2;** the final solution is

$$y = 0{\cdot}858952 + 0{\cdot}108828\, T_2(x) + 0{\cdot}030184\, T_4(x) + 0{\cdot}001757\, T_6(x) + 0{\cdot}000263\, T_8(x) + 0{\cdot}000014\, T_{10}(x) + 0{\cdot}000002\, T_{12}(x) + \ldots \quad (12)$$

The correct solution of (11) is $y = \exp\{\frac{1}{4}(x^4 - 1)\}$, so that $y(0) = 0{\cdot}778801$; the value of $y(0)$ obtained from the series (12) is 0·778802.

Eigenvalue problems

Since any linear differential equation can be transformed into an infinite set of simultaneous equations in the coefficients a_r, any linear eigenvalue problem can similarly be transformed into a determinantal equation of infinite dimensions. In general this equation can be solved approximately by ignoring all except the first n

Table 3

Steps in the solution of the eigenvalue problem $y'' + \lambda(x+1)y = 0, \quad y(1) = y(-1) = 0$

k	$\Lambda = 1/\lambda$	a_2	a_3	a_4	a_5	a_6	a_7	a_8
0		1						
1	0·417	1	0·17	−0·05	−0·02			
2	0·4231	1	0·152	−0·048	−0·021	−0·001	0·001	
3	0·42185	1	0·15316	−0·04816	−0·02090	−0·00072	0·00065	0·00012
4	0·42203	1	0·15304	−0·04814	−0·02087	−0·00073	0·00065	0·00012
5	0·42202							

rows and columns of the determinant. When, however, the eigenvalue problem involves a differential equation of the form

$$y'' + \lambda P(x)y = 0$$

a more powerful method of solution is available, based on the iterative procedure already described. This type of problem will therefore be considered in detail.

Suppose first that the boundary conditions are of the form $y(-1) = y(1) = 0$. With the same notation as before, it follows immediately from equation (10) that

$$\lambda \sum_{s=2}^{\infty} (U_{r-1,s} - U_{r+1,s})a_s + 8ra_r = 0, \quad r \geqslant 2.$$

This may be written as

$$\sum_{s=2}^{\infty} b_{rs}a_s = \Lambda a_r, \quad r \geqslant 2,$$

where $$b_{rs} = \frac{1}{8r}(U_{r+1,s} - U_{r-1,s})$$

and $$\Lambda = 1/\lambda.$$

The eigenvalues λ of the differential equation are therefore given immediately by the latent roots Λ of the infinite matrix $[b_{rs}]$; and the determination of the fundamental eigenvalue (which is often the only one required) is equivalent to the evaluation of the largest latent root of the matrix. The iterative method suggested below for determining the fundamental eigenvalue is in fact a slight modification of the familiar iterative procedure for calculating a dominant latent root.

In general a_2 may be expected to be the largest of the a_rs when a_0 and a_1 have been eliminated; it is convenient therefore to take a_2 as unity throughout. (A few obvious changes will be necessary if a_2 vanishes, e.g. if y is an odd function of x.) The iterative procedure is then defined by the formulae

$$\left.\begin{aligned} &\Lambda^{(k)} = \sum_{s=2}^{\infty} b_{2s}a_s^{(k-1)} \\ &a_2^{(k)} = 1 \\ &(\Lambda^{(k)} - b_{rr})a_r^{(k)} = \sum_{s=2}^{r-1} b_{rs}a_s^{(k)} + \sum_{s=r+1}^{\infty} b_{rs}a_s^{(k-1)}, \\ &\qquad\qquad\qquad\qquad r \geqslant 3. \end{aligned}\right\} \quad (13)$$

It will usually be convenient to start with $a_2^{(0)} = 1$, and $a_r^{(0)} = 0$ for $r \geqslant 3$. As before as many a_rs as are significant are retained at each stage of the iteration.

As an example, consider the equation

$$y'' + \lambda(x+1)y = 0, \quad y(1) = y(-1) = 0. \quad (14)$$

The steps in the solution are set out in **Table 3.** It will be seen that $\Lambda = 0{\cdot}42202$, so $\lambda = 2{\cdot}3696$; the correct fundamental eigenvalue of (14) is in fact $2{\cdot}36953$. The corresponding solution of the differential equation is

$$\begin{aligned} y = A[&-0{\cdot}95125 - 0{\cdot}13282\, T_1(x) + T_2(x) \\ &+ 0{\cdot}15304\, T_3(x) - 0{\cdot}04814\, T_4(x) \\ &- 0{\cdot}02087\, T_5(x) - 0{\cdot}00073\, T_6(x) \\ &+ 0{\cdot}00065\, T_7(x) + 0{\cdot}00012\, T_8(x) \ldots]. \end{aligned}$$

The above method can also be used when the boundary conditions involve y' as well as y. For differential equations of the form

$$y'' + \lambda P(x)y = 0,$$

equation (8) can be written in the form

$$\begin{aligned} \frac{1}{8r}\sum_{s=0}^{\infty}{}' \Bigg[&\frac{1}{r+1}(\pi_{r+s+1} + \pi_{r-s+1}) \\ &- \frac{1}{r-1}(\pi_{r+s-1} + \pi_{r-s-1})\Bigg]a_s = \Lambda a_r \quad (15) \end{aligned}$$

where $\Lambda = 1/\lambda$ as before. The two boundary conditions may be used to eliminate a_0 and a_1 from (15), so that it reduces to the form

$$\sum_{s=2}^{\infty} b_{rs}a_s = \Lambda a_r, \quad r \geqslant 2.$$

The values of Λ and a_r can then be determined by the iterative procedure (13).

For example, for the equation

$$y'' + \lambda y = 0, \quad y(1) = 0, y'(-1) = 0,$$

equation (15) reduces to

$$-\frac{a_{r-2}}{4r(r-1)} + \frac{a_r}{2(r^2-1)} - \frac{a_{r+2}}{4r(r+1)} = \Lambda a_r,$$

Table 4

Steps in the solution of the eigenvalue problem $y'' + \lambda y = 0, y(1) = 0, \; y'(-1) = 0$

k	$\Lambda = 1/\lambda$	a_2	a_3	a_4	a_5	a_6
0		1				
1	1·42	1	−0·17	−0·02		
2	1·67	1	−0·125	−0·013	0·001	
3	1·60	1	−0·1346	−0·0133	0·0011	0·0001
4	1·622	1	−0·1323	−0·0131	0·0010	0·0001
5	1·621	1	−0·13270	−0·01312	0·00104	0·00007
6	1·6213	1	−0·13260	−0·01312	0·00104	0·00007
7	1·6211					

and the initial conditions are given by

$$\tfrac{1}{2}a_0 + a_1 + a_2 + a_3 + a_4 + \ldots = 0$$
$$a_1 - 4a_2 + 9a_3 - 16a_4 + \ldots = 0.$$

The iterative procedure is therefore

$$24\Lambda^{(k)} = 34 - 48a_3^{(k-1)} + 101a_4^{(k-1)} - 144a_5^{(k-1)} + 222a_6^{(k-1)} - 288a_7^{(k-1)} + \ldots$$

$$a_2^{(k)} = 1$$

$$(48\Lambda^{(k)} - 21)a_3^{(k)} = -8 - 32a_4^{(k-1)} + 49a_5^{(k-1)} - 72a_6^{(k-1)} + 98a_7^{(k-1)} - \ldots$$

$$2r[2(r^2 - 1)\Lambda^{(k)} - 1]a_r^{(k)} = -(r+1)a_{r-2}^{(k)} - (r-1)a_{r+2}^{(k-1)}, \quad r \geqslant 4.$$

The steps in the solution are set out in **Table 4.** It will be seen that $\Lambda = 1\cdot6211$, so $\lambda = 0\cdot6169$; the correct value of λ is, of course, $\pi^2/16 = 0\cdot61685$.

In order to find eigenvalues other than the fundamental, it is necessary to eliminate the dominant latent root from the matrix $[b_{rs}]$; the method of doing this has been well described in *Modern Computing Methods* (1961), page 26, and need not be repeated here. When this root has been removed, the next eigenvalue may be found by an iterative procedure similar to that described above.

Conclusions

It has been shown that Chebyshev expansions may be used to find the numerical solutions of a general class of linear differential equations; but the method appears to be particularly advantageous for the following three types of problem:

(*a*) First-order equations of the form $y' + P(x)y = F(x)$, where $P(x)$ is of order 1 or smaller, and systems of such equations.

(*b*) Second-order equations of the form $y'' + P(x)y = F(x)$, especially when the boundary conditions specify values of y only.

(*c*) Eigenvalue problems involving differential equations of the form $y'' + \lambda P(x)y = 0$.

For each of these types of problem, iterative procedures are available which are particularly suitable for automatic computation.

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