Accelerated convergence of numerical solution of linear and non-linear vector field problems

By S. V. Ahamed*

The numerical solution of vector field problems on digital computers is a slow process, especially when the characteristics of the regions investigated vary considerably. A method for the acceleration of the convergence of iterative procedures is described and applied to linear and non-linear problems. Physical considerations based on Stokes' theorem are utilized to modify Southwell's relaxation technique.

Linear and non-linear fields are common to electromagnetic, electrostatic, fluid-flow and heat-flow problems. The literature on numerical solution of electromagnetic problems is not abundant. When the iron regions are ascribed the usual permeabilities of ferromagnetic materials, the numerical solutions converge extremely slowly, necessitating prohibitive computer time. Furthermore, the solutions often become unstable if the permeabilities in the iron region are calculated from non-linear algebraic relations representing the magnetization characteristics. Recently, Mamak and Laithwaite (1960) have solved a few simplified electromagnetic field problems on computers. Like most of the authors before, they have avoided the problems of slow convergence and poor stability by assuming the permeability of iron as infinite.

A method that accelerates the convergence of both linear and non-linear vector field problems has been developed in this paper. The method is illustrated with electromagnetic field problems. When modified, it is readily applicable for the acceleration of convergence of the iterative solutions of other field problems for which Stokes' theorem is valid and can be stated by Poisson's or a non-linear equivalent of Poisson's partial differential equation.

The partial differential equation in two dimensions

The basic equation governing the distribution of vector potential in a non- linear quasi-Poissonian field in which permeabilities depend on a field quantity is

$$\frac{\partial}{\partial x}\left(\frac{1}{\mu}\frac{\partial A}{\partial y}\right)+\frac{\partial}{\partial y}\left(\frac{1}{\mu}\frac{\partial A}{\partial y}\right)=J. \qquad (1)$$

The finite-difference form of equation (1) for a mesh of **Fig. 1** can be shown to be

$$A_0=\frac{J_0+\dfrac{A_1\alpha_1}{\mu_1}+\dfrac{A_2\alpha_2}{\mu_2}+\dfrac{A_3\alpha_3}{\mu_3}+\dfrac{A_4\alpha_4}{\mu_4}}{\dfrac{\alpha_1}{\mu_1}+\dfrac{\alpha_2}{\mu_2}+\dfrac{\alpha_3}{\mu_3}+\dfrac{\alpha_4}{\mu_4}} \qquad (2)$$

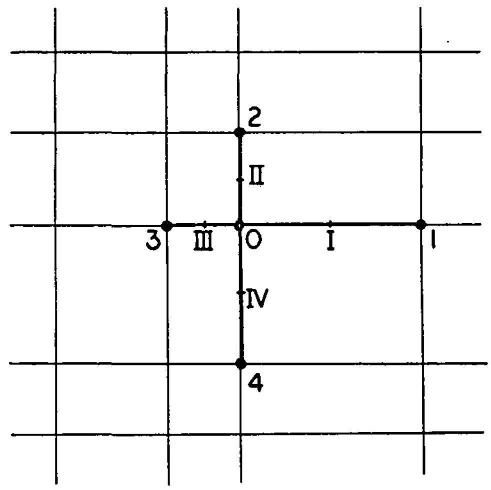

Fig. 1.—A typical non-uniform mesh point

where

J_0 = current density vector assumed uniform in the area containing the point 0

A_n = vector potential at n (n is 1, 2, 3, 4)

$\mu_{1,2,3,4}$ = permeabilities at the centre of finite distances 01, 02, 03 and 04, respectively.

The mesh constants α_n are

$$\alpha_1=\frac{2}{h_1(h_1+h_3)},\quad \alpha_2=\frac{2}{h_2(h_2+h_4)} \qquad 3(a), (b)$$

$$\alpha_3=\frac{2}{h_3(h_1+h_3)},\quad \alpha_4=\frac{2}{h_4(h_2+h_4)} \qquad 3(c), (d)$$

where

$h_{1,2,3,4}$ = 01, 02, 03 and 04, respectively in Fig. 1.

The modified relaxation method

In the numerical solution of the partial differential equations, using relaxation methods suggested by Southwell (1946) and Allen (1954), equation (2) is written

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for each mesh point. The convergence of the relaxation method is reasonable for problems in which the permeabilities of individual regions do not vary greatly. In electromagnetic field problems the permeabilities of two adjoining regions like iron and air vary considerably, and the use of relaxation methods results in an extremely slow convergence. In non-linear problems where the permeability is a function of magnetic induction, successive iterates begin to oscillate about the true solution, and in some cases this leads to divergence of the iterative process.

The application of Stokes' theorem to Maxwell's equation leads to the well-known Ampere's law

$$\oint_c \bar{H} \, . \, d\bar{l} = \int\int_s \bar{J} \, . \, d\bar{s} \tag{4}$$

that must also be satisfied in the field. This relation, equation (4), is utilized as the basis of the new method to adjust the vector potentials between two iterations. After each iteration the line integral of the magnetic intensity around a suitably chosen closed path in the region is evaluated and denoted F. The ratio of the net current enclosed by the path of integration and of the value of the line integral after the nth iteration, F_n, is denoted c_n. Equation (4) is not generally satisfied at the start of the iteration procedure, and the ratio c_n will not be unity. The vector potentials over the entire regions are multiplied by c_n in order to satisfy equation (4). The process of iterations and multiplication of vector potentials is carried out until

$$0{\cdot}999 < c_n < 1{\cdot}001. \tag{5}$$

If the value of c_n falls in the range of inequality (5) then the multiplication of vector potential is discontinued, and plain relaxation is carried out for a few more iterations. At this stage one can assume that the iterations have converged sufficiently close to the solution.

Explanation of modified relaxation method

Equation (2) is derived from the fundamental equation (1). When equation (2) is applied at every grid point many times, as in Southwell's relaxation methods, the integral form of equation (1) written as equation (4) is also satisfied. Each time the iteration procedure is carried out, the magnetic induction at the centre of the mesh distances 01, 02, 03, and 04 in Fig. 1 changes in the appropriate direction, making the right side of equation (4) closer to its left side. This change of magnetic induction is more effective in the modified relaxation method, which is a special form of block relaxation (1956). The magnetic induction over the entire area is modified by multiplication of all the vector potentials by the scalar constant c_n. This constant assumes a value greater than unity when the value of the right side of equation (4) becomes greater than its left side, and hence increases the magnetic induction in the entire region. Conversely the constant assumes a value less than unity when the value of the right side of equation (4) becomes less than its left side, and hence decreases the magnetic induction in the region. When the constant becomes unity, the multiplication of the vector potentials by c_n has no effect, and an equilibrium condition is reached. In local regions the repeated application of equation (2) is adequate to govern the distribution of flux densities and the satisfaction of equation (4) in those regions.

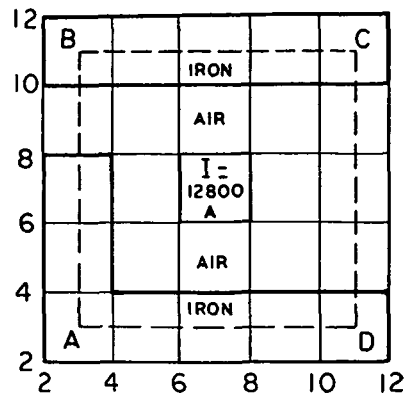

Fig. 2(a).—A schematic electromagnet

Variation of permeability with saturation

In non-linear electromagnetic problems, when the constant c_n is greater than unity, the permeabilities decrease due to the increased value of flux densities. The decrease in the value of permeabilities generally increases the line integral of the magnetic intensity, H, far beyond the magnitude of enclosed current. This makes the new value of c_n less than unity. The inverse effect of c_n on permeability causes undamped oscillations. To avoid these oscillations and to stabilize the solution, an under-relaxation of permeabilities is desirable. The new permeabilities at any mesh point in the non-linear region will then be equal to their old value plus a fraction of the calculated change. Experience has shown that if the actual change in permeabilities is 10 to 15% of the calculated change, the computer solution exhibits good stability and does not need many iterations to reach the final solution.

Application of the method and results

Fig. 2(a) shows half of the cross section of a schematic electromagnet. It consists of an air region and two iron regions. A uniform mesh with 36 points is chosen for sample analysis. The relative permeability of iron was assumed to be 1000.

The number of iterations required to reach sufficient convergence (when the line integral of magnetic intensity just equals the current enclosed) is shown in **Fig. 2(b)** by ordinary relaxation method (curve A). The number of iterations by the modified method is shown by curve B.

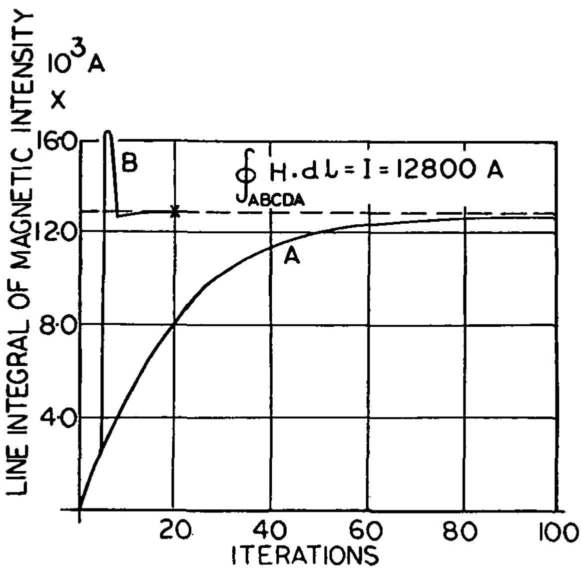

Fig. 2(b).—Comparison of convergence

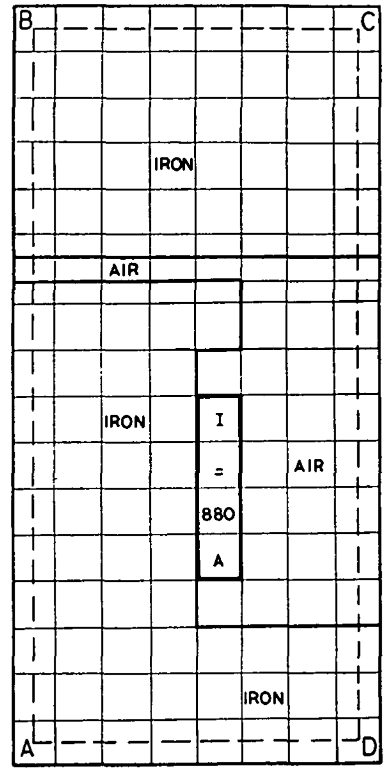

Fig. 3(a).—Cross-section of a salient pole of a synchronous generator

Fig. 3(a) depicts another sample problem representing a synchronous generator. The field distribution has been found by using equation (2), first by over-relaxation method and then by the modified method. An irregular mesh with 153 points was chosen. The relative permeability of iron was 15,600. The line integral of magnetic intensity around ABCDA (Fig. 3(*a*)) was plotted for the various number of iterations for the over-relaxation method (over-relaxation factor = 1·7) in **Fig. 3(b)**, curve A; and for the modified relaxation method, curve B.

The method applied to non-linear Poissonian field is shown in **Fig. 4.** The permeability in the iron of Fig. 3(*a*) is a non-linear function of the magnetic induction. The non-linear relation of the magnetic intensity *H* and of the magnetic induction *B* was chosen to represent the experimental saturation curve. The results of the over-relaxation method (curve A, Fig. 4) exhibits oscillations that were more severe than the oscillations of the linear case shown by curve A, Fig. 3(*b*). In a majority of the cases with larger oscillations the solutions became unstable and diverged.

The results obtained by the modified relaxation method have neither exhibited large oscillations nor any tendency to diverge. This was true even for problems that required a large number of mesh points. Experience has shown that problems with about five thousand mesh points have converged satisfactorily within 200 iterations.

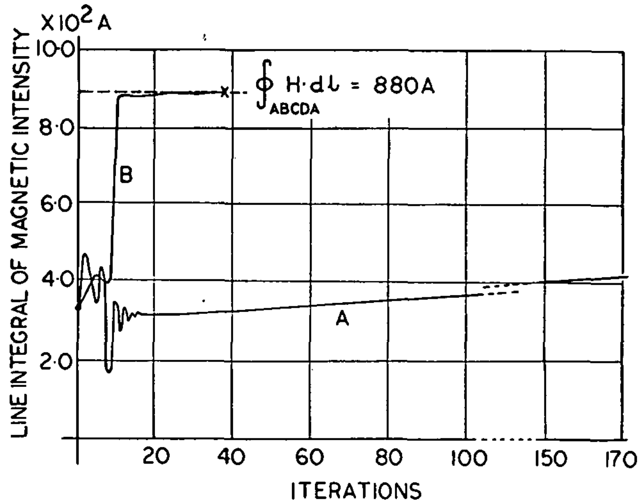

Fig. 3(b).—Comparison of convergence

Conclusions

Figs. 2(*b*), 3(*b*) and 4 prove that the number of iterations necessary for convergence of the modified relaxation method proposed here is a fraction of the number of iterations necessary by usual relaxation methods. This results in a considerable saving of computer time, especially when the number of grid points is large. With non-linear vector potential problems the method eliminates divergence and instability of results. It accelerates considerably the rate of convergence.

The method, though illustrated for the vector potential in electromagnetic field is generally applicable to many linear and non-linear vector field problems. The principle may be extended to the electrostatic field where the electric intensity, due to net charges, is present. In these problems Gauss' theorem applies.

One has then to evaluate the net value of the surface integral of the displacement and equate it to the enclosed charge. Problems in hydrodynamics and heat flow may be solved by this method.

The method developed here is generally applicable to field problems which are

(*a*) not "curl free," where Stokes' theorem applies,
(*b*) not "source free," where Gauss' theorem applies.

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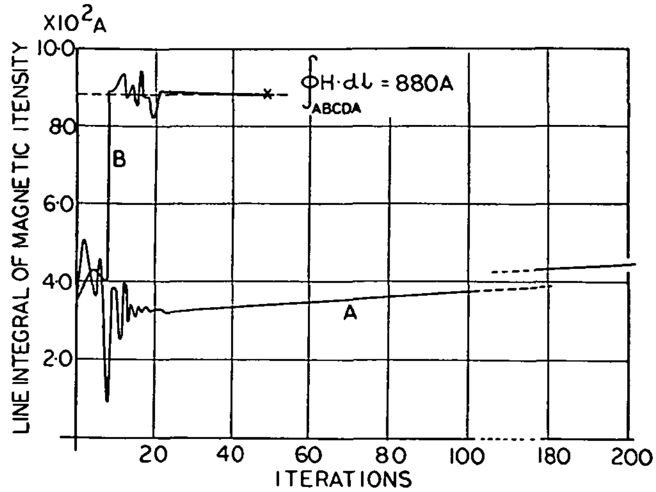

Fig. 4.—Comparison of convergence for a non-linear Poissonian field in iron regions of Fig. 3(a)

References

Allen, D. N. De G. (1954). *Relaxation Methods*, New York: McGraw-Hill Book Company.

Arms, R. J., Gates, L. D., and Zondeck, B. (1956). "A Method of Block Iteration," *Journal of the Society for Industrial Applied Mathematics*, Vol. 4, pp. 220–229.

Mamak, R. S., and Laithwaite, E. L. (1960). "Numerical Evaluation of Inductances and A.C. Resistances," *Proc. IEE*, Part C, Vol. 108, pp. 240–246.

Southwell, R. V. (1946). *Relaxation Methods in Engineering and Science*, Oxford: The Clarendon Press.

Book Reviews (*Continued from p.* 20)

Whilst the author does not claim to have covered every possible application, the coverage is nevertheless comprehensive, and the material proceeds in a logical way. Many references are given on specific topics for the specialized reader.

Altogether a very readable and informative book which should be an asset to the circuit engineer.

J. C. Vickery

Reliable Computation in the Presence of Noise, by S. Winograd and J. D. Cowan, 1964; 96 pages. (Cambridge, Massachusetts: *M.I.T. press*, 38s.)

This is one of the series of *Research Monographs* issued by the *M.I.T. Press*. These monographs permit the presentation of research in a more detailed way than is reasonably possible in a scientific journal whilst, at the same time, obtaining earlier publication than would be possible in a standard text-book.

The purpose of this particular publication is to extend Shannon's noisy-channel coding theorem to include the case of computation with noisy modules rather than communication. It then continues to show how error-correcting codes may be employed in the construction of reliable automata from less reliable modules.

The first two chapters give, in eighteen pages, an introduction to the relevant aspects of information theory and the theory of automata. This is followed by a discussion of the work of Von Neumann and others on the reliability of automata, which is then extended in the next chapter to computation with noisy modules. It is shown that such a system may be decomposed into an error-free computation module and a noisy communication channel.

Chapters 6 and 7 contain various arguments leading to the conclusion that it is wrong to consider module networks as separable into encoding, computing and decoding networks in which encoding and decoding are free from error. Chapter 8 therefore considers only noisy modules at each stage, and describes the construction of networks of varying degrees of reliability. These designs depend on the assumption that the probability of modular malfunction is independent of modular complexity.

The final chapter shows that synaptic errors may be incorporated, the effect of such errors being controlled by the use of networks of still greater redundancy. This chapter also discusses the effect of errors of connection in the redundant networks. There is a short appendix followed by a list of about fifty references.

The formal arguments are set out very clearly and they are well illustrated by numerous network diagrams which are beautifully reproduced. Whilst the detailed sections of the book are for the specialist, the general arguments and conclusions are of interest to anyone working in the field of computers.

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