

$$\begin{aligned} \|R\|_2 &\leq \|S\|_2 \|G_s^T\|_2 \|(F_s - a_{nn}^{(s)}I)^{-1}\|_2 \\ &\leq \epsilon \max |\lambda_i' - a_{nn}^{(s)}|^{-1} \\ &< 2\epsilon/\delta \text{ from (22)}. \end{aligned} \tag{36}$$

Hence from equation (27)

$$\begin{aligned} \|Z\|_2 &\leq \|R\|_2 \|G_s\|_2 + \|S\|_2 \|H_s - (a_{nn}^{(s)})I\| \\ &\leq 2\epsilon^2/\delta + \max |\lambda_i' - a_{nn}^{(s)}| \\ &< 2\epsilon^2/\delta + 3\epsilon^2/\delta \\ &= 5\epsilon^2/\delta. \end{aligned} \tag{37}$$

Finally from equation (29)

$$\|G_{s+1}^T\|_2 \leq \|Z\|_2 \|Q^T\|_2 = \|Z\|_2 \|R\|_2 < 10\epsilon^3/\delta^2. \tag{38}$$

In connexion with the symmetric  $LL^T$  algorithm for positive definite matrices Rutishauser (1960) has described a method of choosing  $k_s$  which gives cubic convergence to the smallest eigenvalue. The proof we have just given may be modified to show that Rutishauser's technique gives cubic convergence whatever the multiplicity of the smallest eigenvalue. However, when this eigenvalue is of multiplicity  $r$  Rutishauser's technique requires the solution of an eigenvalue problem of order  $r$  and is therefore not quite so convenient as the process we have just described.

### Numerical example

In Table 1 we exhibit a matrix  $A_s$  of order four with the eigenvalues 6, 4, 2, 2, at a late stage in the

### References

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 RUTISHAUSER, H. (1958). "Solution of eigenvalue problems with the LR transformation," *U.S. Nat. Bur. Stand. Appl. Math. Ser.*, Vol. 49, pp. 47-81.  
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Table 1

Lower triangle of $A_s$			
5.81522 813			
0.57853 605	4.18421 247		
-0.03866 598	-0.00586 326	2.00039 187	
-0.00506 679	-0.01912 924	0.00005 135	2.00016 753
Lower triangle of $A_{s+1}$			
5.95048 519			
0.31076 986	4.04951 481		
-0.00000 168	0.00000 026	2.00000 000	
-0.00000 022	-0.00000 145	0.00000 000	2.00000 000

iterative process. The  $2 \times 2$  matrix  $G_s^T$  is such that  $\|G_s^T\|_2$  is of the order of 0.04. Since  $\delta = 2$  the condition  $\epsilon < \frac{1}{3}\delta$  is certainly satisfied so that  $H_s - 2I$  must certainly be small; in fact it will be seen that it is of the order of 0.0004. One iteration was performed using  $k_s = a_{44}^{(s)}$  and the resulting  $A_{s+1}$  is displayed. It will be seen that  $\|G_{s+1}^T\|_2 < 0.000002$ , while  $H_{s+1} = 2I$  to working accuracy.

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## Book Reviews

*Mathematical Theory of Automata*, edited by JEROME FOX, 1964; 640 pages. (London and New York: John Wiley and Sons., 75s.).

This book is the proceedings of a symposium held in New York in April, 1962. The thirty-three papers are almost all reports of research done by the authors and not reviews or expository papers; they are therefore at a high level of specialization. This is not a book for the general reader, but it contains a number of interesting papers for reference by experts in this field. No formal grouping into subjects has been made though the papers divide into fairly clear-cut sections: theorem proving, computability, finite-state machines and self-organizing systems. Those who work in these areas will wish to have access to the book; it is unlikely to be useful to others.

A few of the papers can be read by the non-specialist with little prior knowledge. A review paper by Davis gives a short and lucid account of some of the classic unsolvable

problems showing the reasons why the problems arose and their importance. Gelernter describes the methods used in his heuristic Geometry Theorem Machine to set up theorems and sub-theorems for the machine to prove. A method of pattern recognition is described by Unger which uses a two-dimensional grid of combinational cells to recognize features of patterns such as concavity to the right or the presence of holes.

The papers on finite-state machines are made difficult to read by the absence of a unified system of notation. A number of mathematical disciplines have contributed to this subject; the theory of groups and of semi groups, lattice theory, graph theory and the theory of computability to mention some examples, and the result seems to have been that authors have invented cumbersome notations which are difficult to read and remember.

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