Evaluation of certain definite integrals frequently encountered in radiation and diffraction problems involving circular geometry*

By E. J. Martin, Jr.† and Peter C. Patton§

Consideration is given to the evaluation of definite integrals of the form

$$I = \int_0^{2\pi} \cos n(\xi + \psi_n)[\exp(\pm i\beta R)/R] \begin{matrix}\sin \\ \cos\end{matrix} \xi d\xi$$

where n is any integer and $R = [r^2 + a^2 - 2ra \sin\theta \cos\xi]^{1/2}$. These integrals are encountered in a variety of electromagnetic radiation and diffraction problems involving circular geometry.

Various techniques that have been used to obtain approximate evaluations of these field integrals are discussed briefly. Subsequently, a rather straightforward analytical method is described which gives closed-form evaluation of the integrals without resorting to any simplifying approximations. The results of this method are presented in terms of near-zone or source-region functions, depending on whether $r > a$ or vice versa. These functions are defined by infinite series which involve well-known mathematical functions.

Some interesting properties of the near-zone and source-region functions are considered, and it is shown that their infinite series representations are amenable to machine computation. Problems associated with the calculation of numerical values are discussed and generalized algorithms for the near-zone and source-region functions are presented.

1. Introduction

Recent advances in radio-frequency technology have resulted in a growing need for accurate knowledge about the total electromagnetic fields that are produced in the immediate vicinity of current distributions on antennas or diffracting objects. Analytical methods, based largely on approximation techniques, that are currently used to calculate these fields, do not provide the degree of accuracy that is required in many instances. Of particular interest in this connection are the definite integrals of the form

$$I = \int_0^{2\pi} \cos n(\xi + \psi_n)[\exp(\pm i\beta R)/R] \begin{matrix}\sin \\ \cos\end{matrix} \xi d\xi \quad (1)$$

where n is any integer and

$$R = [r^2 + a^2 - 2ra \sin\theta \cos\xi]^{1/2}. \quad (2)$$

These integrals are encountered in a variety of electromagnetic radiation and diffraction problems involving circular geometry. In the past, a number of diffeıent approximation techniques have been used to obtain evaluations of these integrals.

What is probably the most widely used approach to the approximate evaluations of the integrals (1) begins with the removal of a factor r from the radical on the right side of (2). Next, the remaining square root is expanded into a power series in the quantity a/r. An attempt is made to achieve an accuracy commensurate with the requirements of the problem under consideration by selecting the number of terms of the power series expansion that are retained in subsequent calculations. This technique forms the basis of both the so-called "far-zone" and the "Fresnel-region" approximations.

According to the "far-zone" approximation, only terms of the power series expansion of (2) that are of order *zero* in a/r are used in expressing the amplitude factor, $1/R$, that appears in the integrand of (1). However, terms of both orders *zero* and *one* are used in expressing the phase factor, $\exp(\pm i\beta R)$. This approximation has been widely applied to the solution of problems concerning the calculation of the radiation fields produced at great distances from structures such as the thin-wire circular loop antenna (Foster, 1944; Wait, 1959a; Martin, 1960), the narrow annular slot

* The research reported herein was accomplished without sponsorship from any tax-supported agency of the U.S. Government.

† *Engineering Division, Midwest Research Institute, Kansas City, Missouri.*

§ *Univac Division, Remington Rand Limited,* 65, *Holborn Viaduct, London, E.C.* 1. *Presently Leiter der Rechengruppe, Institut für Statik und Dynamik der Luft- und Raumfahrtkonstruktionen, Technische Hochschule, Stuttgart, Germany.*

antenna (Wait, 1959b), open-ended coaxial lines (Ramo and Whinnery, 1947), and cylindrical radiators of finite (circular) cross section (King, 1964).

The more accurate "Fresnel-region" approximation also depends on the assumption that $1/R \approx 1/r$, but retains power series terms up to the order *two* in r for purposes of expressing the phase factor in the integrand of (1). This approximation has found extensive application in the calculation of the so-called optical-Fresnel fields produced by illuminated apertures (Silver, 1939). A variation on this second-order approximation, obtained by means of an iterative process, has been developed by Hu (Hu, 1960) in connection with the investigation of the Fresnel-region field distributions of circular aperture antennas.

A different approach to the development of an approximation technique has been used by Seshardi and Wu (1960; 1963) in the evaluation of special cases of (1) that are encountered in the analysis of the diffraction of electromagnetic waves by circular apertures in infinite screens having specific conductive properties. This approximation is obtained by means of a rather lengthy derivation based on the fact that, under the assumed conditions, significant contributions to the integral correspond to only two stationary values of ξ. A somewhat similar technique has been employed by Lavine and Papus (1951) in the investigation of circular diffraction antennas, and by Galejs and Thompson (1962) in the calculation of admittances of cavity-backed annular slots.

A method has been developed recently for obtaining "closed-form" evaluations of the integrals (1) without resorting to any simplifying approximations (Martin, 1963; 1964). This exact evaluation technique is based on a rather straightforward analytical procedure that involves only relatively simple mathematics. The procedure begins with the expansion of the integrand into a Jacobi series expressed in terms of spherical Bessel functions in βa and βr, associated Legendre functions in $\cos\theta$, and simple trigonometric functions in ξ. The exact form of the Jacobi series depends on the particular case of (1) that is under consideration but, in all cases, term-by-term integration with respect to ξ is a simple matter. Rearrangement of the integrated series gives an evaluation of the integral in terms of the "near-zone functions," $F^{(\pm)}_{n\pm1}(a,r,\theta)$ in those cases where $r > a$ or the "source-region functions," $G^{(\pm)}_{n\pm1}(a,r,\theta)$ when $r < a$. Each of these classes of functions is generally defined by an infinite series in a form that is amenable to machine computation.

The validity of this evaluation technique has been confirmed theoretically by demonstrating that the series expansions of the integrands are uniformly convergent on the interval $0 \leqslant \xi \leqslant 2\pi$. A practical "proof" of the results has been obtained by showing that the exact expressions which are obtained by this method actually contain a number of previously derived approximate results as special or limiting cases.

This paper is devoted to a brief summary of the analytical techniques used to obtain exact evaluations of the integrals (1), and to the development of a generalized computer program for machine calculation of numerical values for the "near-zone" and "source-region" functions that are generated by the analysis.

2. Summary of analytical technique

In order to demonstrate the analytical procedure involved in obtaining exact, closed-form evaluations of the integrals (1), it will be convenient to limit consideration temporarily to that particular case, of the four that are included in (1), which is defined by

$$I_N^{(+)} = \int_0^{2\pi} \cos n\,(\xi + \psi_n)[\exp\,(i\beta R)/R] \begin{matrix}\sin\\ \cos\end{matrix}\,\xi d\xi, \qquad (3)$$

with the added stipulation that $r > a$. Under these circumstances, the exponential factor appearing in the integrand can be expanded in terms of Bessel functions and Legendre polynomials as (Erdelyi, 1955a)

$$\exp(i\beta R)/R = i\beta \sum_{m=0}^{\infty} (2m+1) j_m(\beta a) h_m^{(1)}(\beta r) P_m(\sin\theta\cos\xi), \qquad (4)$$

where the spherical Bessel function notation has been used to replace the cylindrical Bessel function notation according to the general relation (Stratton, 1941)

$$z_m(x) = (\pi/2x)^{1/2} Z_{m+1/2}(x).$$

Moreover, the Legendre polynomials, $P_m(\sin\theta\cos\xi)$ can be expanded in terms of Associated Legendre functions (Erdelyi, 1955b) to put (4) into the form

$$\exp\,(i\beta R)/R = i\beta \sum_{m=0}^{\infty} (2m+1) j_m(\beta a) h_m^{(1)}(\beta r) \Big[P_m(0) P_m\,(\cos\theta) + 2 \sum_{k=1}^{m} \frac{(m-k)!}{(m+k)!} P_m^k(0) P_m^k\,(\cos\theta) \cos k\xi \Big]. \qquad (5)$$

Now, in the physical problems in which (3) is encountered, both a and r are real, positive quantities while θ and ξ are real angles. Therefore,

$$2ar\sin\theta\cos\xi < r^2 + a^2$$

and $\exp\,(i\beta R)/R$ is analytic for $0 \leqslant \xi \leqslant 2\pi$. It follows (Szego, 1959) that the Jacobi-series expansion given in (5) is uniformly convergent over the interval of ξ that is of interest. Consequently, (5) may be substituted into (3), and integration of the resulting expression may be carried out term-by-term as indicated by

$$I_N^{(+)} = i\beta \sum_{m=0}^{\infty} (2m+1) j_m(\beta a) h_m^{(1)}(\beta r) \Big[P_m(0) P_m(\cos\theta) \int_0^{2\pi} \cos n(\xi + \psi_n) \begin{matrix}\sin\\ \cos\end{matrix}\,\xi d\xi + 2 \sum_{k=1}^{m} \frac{(m-k)!}{(m+k)!} P_m^k(0) P_m^k(\cos\theta) \int_0^{2\pi} \cos n(\xi + \psi_n) \cos k\xi \begin{matrix}\sin\\ \cos\end{matrix}\,\xi d\xi \Big]. \qquad (6)$$

Evaluation of the simple integrals appearing on the right side of (6) is facilitated by the orthogonal properties of the trigonometric functions. After appropriate manipulation of the expressions that are obtained from this process, it is found that

$$I_N^{(+)} = \int_0^{2\pi} \cos n(\xi + \psi_n)[\exp(i\beta R)/R] \begin{matrix}\sin\\ \cos\end{matrix} \xi\, d\xi;\ r > a$$

$$= \mp\, i\pi\beta \begin{matrix}\sin\\ \cos\end{matrix} n\psi_n [F_{n-1}^{(+)}(a, r, \theta) \mp F_{n+1}^{(+)}(a, r, \theta)], \quad (7)$$

where the "near-zone" functions, $F_{n\pm1}^{(+)}(a, r, \theta)$ are defined, in general, by

$$F_n^{(+)}(a, r, \theta) = \sum_{m=0}^{\infty} \frac{(-1)^{m+n}(4m+2n+1)(2m)!}{(2)^{2m+n}(m)!\,(m+n)!} j_{2m+n}(\beta a) h_{2m+n}^{(1)}(\beta r) P_{2m+n}^n(\cos\theta). \quad (8)$$

The foregoing results can be extended to the second particular case of (1) where the exponential factor is still $\exp(i\beta R)/R$ but now $r < a$. This is readily accomplished by simply interchanging the arguments of the two Spherical Bessel functions appearing in (4) and noting that the obvious result of this change is

$$I_s^{(+)} = \int_0^{2\pi} \cos n(\xi + \psi_n)\,[\exp(i\beta R)/R] \begin{matrix}\sin\\ \cos\end{matrix} \xi d\xi;\ r < a$$

$$= \mp\, i\pi\beta \begin{matrix}\sin\\ \cos\end{matrix} n\psi_n [G_{n-1}^{(+)}(a, r, \theta) \mp G_{n+1}^{(+)}(a, r, \theta)], \quad (9)$$

where the "source-region" functions, $G_{n\pm1}^{(+)}(a, r, \theta)$ are defined, in general, by

$$G_n^{(+)}(a, r, \theta) = \sum_{m=0}^{\infty} \frac{(-1)^{m+n}(4m+2n+1)(2m)!}{(2)^{2m+n}(m)!\,(m+n)!} h_{2m+n}^{(1)}(\beta a) j_{2m+n}(\beta r) P_{2m+n}^n(\cos\theta). \quad (10$$

The remaining two particular cases of (1) are treated with equal facility by noting that when the exponential factor $\exp(i\beta R)/R$ in (3) is replaced by $\exp(-i\beta R)/R$ the results of the analysis will be the complex conjugates of (7) through (10), respectively. Thus,

$$I_N^{(-)} = \int_0^{2\pi} \cos n(\xi + \psi_n)[\exp(-i\beta R)/R] \begin{matrix}\sin\\ \cos\end{matrix} \xi d\xi;\ r > a$$

$$= \pm\, i\pi\beta \begin{matrix}\sin\\ \cos\end{matrix} n\psi_n [F_{n-1}^{(-)}(a, r, \theta) \mp F_{n+1}^{(-)}(a, r, \theta)], \quad (11)$$

where the "near-zone" functions, $F_{n\mp1}^{(-)}(a, r, \theta)$ are defined, in general, by

$$F_n^{(-)}(a, r, \theta) = \sum_{m=0}^{\infty} \frac{(-1)^{m+n}(4m+2n+1)(2m)!}{(2)^{2m+n}(m)!\,(m+n)!} j_{2m+n}(\beta a) h_{2m+n}^{(2)}(\beta r) P_{2m+n}^n(\cos\theta), \quad (12)$$

while

$$I_s^{(-)} = \int_0^{2\pi} \cos n(\xi + \psi_n)[\exp(-i\beta R)/R] \begin{matrix}\sin\\ \cos\end{matrix} \xi d\xi;\ r < a$$

$$= \pm\, i\pi\beta \begin{matrix}\sin\\ \cos\end{matrix} n\psi_n [G_{n-1}^{(-)}(a, r, \theta) \mp G_{n+1}^{(-)}(a, r, \theta)], \quad (13)$$

where the "source-region" functions, $G_{n\pm1}^{(-)}(a, r, \theta)$ are defined, in general, by

$$G_n^{(-)}(a, r, \theta) = \sum_{m=0}^{\infty} \frac{(-1)^{m+n}(4m+2n+1)(2m)!}{(2)^{2m+n}(m)!\,(m+n)!} h_{2m+n}^{(2)}(\beta a) j_{2m+n}(\beta r) P_{2m+n}^n(\cos\theta). \quad (14)$$

3. Properties of functions

It is of interest to consider some of the properties of the near-zone and source-region functions defined in (8), (10), (12), and (14), and to comment on the significance of these properties:

3.1 *Asymptotic behaviour*

Examination of the behaviour of the near-zone and source-region functions under certain limiting conditions is of importance in establishing relationships between exact evaluations of the field integrals (1) and various approximations. Of particular interest in this connection are the behaviour of the near-zone functions, $F_n^{(\pm)}(a, r, \theta)$ under two different circumstances (viz., when r becomes very large and when a becomes very small), and the behaviour of the source-region functions, $G_n^{(\pm)}(a, r, \theta)$ when r becomes very small.

For the case where r becomes very large,

$$h_{2m+n}^{(1),(2)}(\beta r) \approx (\mp i)^{2m+n+1}[\exp(\pm i\beta r)/(\beta r)]. \quad (15)$$

When this approximation is used in either (8) or (12), and it is recognized that expansion of the cylindrical Bessel functions in terms of spherical Bessel functions gives

$$\sum_{m=0}^{\infty} \frac{(-1)^n(4m+2n+1)(2m)!}{(2)^{2m+n}(m)!\,(m+n)!} j_{2m+n}(\beta a) P_{2m+n}^n(\cos\theta) = J_n(\beta a \sin\theta), \quad (16)$$

it is possible to write

$$F^{(\pm)}(a, r, \theta) \approx (\mp i)^{n+1}[\exp(\pm i\beta r)/(\beta r)] J_n(\beta a \sin\theta), \quad \text{for } (r \to \infty). \quad (17)$$

For the case where a becomes very small, the power-series expansion of spherical Bessel functions gives

$$j_{2m+n}(\beta a) = (2)^{2m+n} \sum_{k=0}^{\infty} \frac{(-1)^k(2m+n+k)!}{(k)!\,(4m+2n+2k+1)!} (\beta a)^{2m+n+2k},$$

from which it can be established that

$$\lim_{a\to0} \{j_{2m+n}(\beta a)/(\beta a)^n\} = \begin{cases}(2)^n(n)!/(2n+1)! & \text{for } m=k=0\\ 0, & \text{otherwise.}\end{cases}$$

Therefore, the near-zone functions become

$$F_n^{(\pm)}(a, r, \theta) \approx (-\beta a)^n h_n^{(1),(2)}(\beta r) P_n^n(\cos\theta)/(2n)!, \quad \text{for } (a \to 0). \tag{18}$$

According to (7) and (11), the special case of (18) where $n = 1$ is of particular importance, because it corresponds to a "uniform distribution" (i.e., $n = 0$) in the integrand of (1). For this special case, (18) becomes

$$F_1^{(\pm)}(a, r, \theta) \approx (\beta a/2) h_1^{(1),(2)}(\beta r) \sin\theta, \text{ for } (a \to 0). \tag{19}$$

It is interesting to note that when $r \to \infty$ in (19), the approximation given in (15) can be used to write

$$F_1^{(\pm)}(a, r, \theta) \approx (-a/2r)\exp(\pm i\beta r)\sin\theta, \text{ for } \begin{pmatrix} a \to 0 \\ r \to \infty \end{pmatrix}.$$

This same result is also obtained from (17) when $n = 1$ and $a \to 0$, since

$$J_1(\beta a \sin\theta) \approx (\beta a \sin\theta)/2, \text{ for } (a \to 0).$$

For the case where $r \to 0$, attention must be focused on the source-region functions, $G_n^{(\pm)}(a, r, \theta)$, rather than the near-zone functions. Under these circumstances, the procedure used to obtain (18) can be applied to (10) and (14) to write

$$G_n^{(\pm)}(a, r, \theta) \approx (-\beta r)^n h_n^{(1),(2)}(\beta a) P_n^n(\cos\theta)/(2n)!, \text{for } (r \to 0).$$

3.2 *Convergence*

Convergence of the infinite series (8), (10), (12) and (14), which define the near-zone and source-region functions, can be demonstrated by means of the ratio test. For example, through the use of the general relation between spherical Bessel functions

$$(2k+1)z_k(x) = x[z_{k-1}(x) + z_{k+1}(x)],$$

it is possible to write the ratio of successive terms of (8) as

$$\mathscr{R} = -\frac{(2m+1)\beta a[j_{2m+n+1}(\beta a) + j_{2m+n+3}(\beta a)] h_{2m+n+2}^{(1)}(\beta r) P_{2m+n+2}^n(\cos\theta)}{(2m+2n+2)(4m+2n+1) j_{2m+n}(\beta a) h_{2m+n}^{(1)}(\beta r) P_{2m+n}^n(\cos\theta)}.$$

Therefore, when m becomes very large $\mathscr{R} \approx (-\beta a)/(2m)$ or

$$\lim_{m \to \infty}[\mathscr{R}] = 0.$$

Clearly, similar results can be obtained from (10), (12) and (14). Consequently, the near-zone and source-region functions are amenable to machine computation.

3.3 *Alternative representations*

The definitions of the near-zone and source-region functions given in (8), (10), (12) and (14) can be put into an alternate form by applying the relations

$$P_{2m+n}^n(0) = \frac{(-1)^{m+n}(2m+2n)!}{(2)^{2m+n}(m)!\,(m+n)!}$$

and

$$P_{m+n}^n(0) = 0, \text{ for } m = 1, 3, 5, \ldots.$$

After convenient alterations of the indices and ranges of summation in the resulting expressions, it is found that

$$F_n^{(\pm)}(a, r, \theta) = \sum_{m=0}^{\infty} \frac{(2m+2n+1)(m)!}{(m+2n)!} P_{m+n}^n(0) j_{m+n}(\beta a) h_{m+n}^{(1),(2)}(\beta r) P_{m+n}^n(\cos\theta), \tag{20}$$

while

$$G_n^{(\pm)}(a, r, \theta) = \sum_{m=0}^{\infty} \frac{(2m+2n+1)(m)!}{(m+2n)!} P_{m+n}^n(0) h_{m+n}^{(1)(2)}(\beta a) j_{m+n}(\beta r) P_{m+n}^n(\cos\theta). \tag{21}$$

The alternate representations of the near-zone and source-region functions given in (20) and (21), respectively, are used as a basis for numerical computations discussed in a subsequent portion of this paper.

3.4 *Interrelationships*

By means of the relation

$$P_{2m+n}^n(\cos\theta) = (-1)^n(2m+2n)!\, P_{2m+n}^{-n}(\cos\theta)/(2m)!$$

(8) can be written as

$$F_n^{(+)}(a, r, \theta) = \sum_{m=0}^{\infty} \frac{(-1)^m(4m+2n+1)(2m+2n)!}{(2)^{2m+n}(m)!(m+n)!} j_{2m+n}(\beta a) h_{2m+n}^{(1)}(\beta r) P_{2m+n}^{-n}(\cos\theta). \tag{22}$$

Changing the index of summation according to the substitution $m + n = p$ converts (22) into the form

$$F_n^{(+)}(a, r, \theta) = \sum_{p=n}^{\infty} \frac{(-1)^{p-n}(4p+2n+1)(2p)!}{(2)^{2p-n}(p)!\,(p-n)!} j_{2p-n}(\beta a) h_{2p-n}^{(1)}(\beta r) P_{2p-n}^{-n}(\cos\theta). \tag{23}$$

However, since $1/(p-n)! = 0$ for $p < n$ the lower limit of summation in (23) can be changed to zero. Then, from a comparison of (8) and (23) it becomes evident that

$$F_n^{(+)}(a, r, \theta) = F_{-n}^{(+)}(a, r, \theta).$$

Clearly, an application of this same procedure to (10), (12) and (14) will establish similar interrelationships for $F_{\pm n}^{(-)}(a, r, \theta)$, $G_{\pm n}^{(+)}(a, r, \theta)$ and $G_{\pm n}^{(-)}(a, r, \theta)$. These interrelationships are currently being investigated along with (20) and (21) in connection with the development of general recurrence relations for the near-zone and source-region functions.

4. An algorithm for the computation of F_n and G_n

Since evaluations of the definite integrals are given in terms of the near-zone and source-region functions

$F_n^{(\pm)}$ and $G_n^{(\pm)}$, a method for obtaining numerical values of these functions is required if the results of the foregoing analysis are to be of any immediate practical value. To this end, a considerable amount of computational experience has been obtained with the series representations (12) and (14), using the IBM 1620 computer, and with (20) and (21), using the UNIVAC 1107 computer. As a result of this experience, it has been possible to develop a generalized computer algorithm for the sets of series expressions representing $F_n^{(-)}$ and $G_n^{(-)}$.

4.1 *Development of the algorithm*

A program was written in ALGOL 60 (Naur, 1960) to prepare a table of values for $F_n^{(-)}$ and $G_n^{(-)}$. ALGOL was considered to be the most suitable high-level computer language available for this task because of its facility for the expression of mathematical algorithms and because there exists an international literature (*Comm. ACM*, Vol. 3, 1960; *The Computer Bull.*, 1964) of ALGOL 60 algorithms for a wide variety of mathematical functions. This algorithm library proved to be helpful in the preparation of the tables for $F_n^{(-)}$ and $G_n^{(-)}$; however, it was found in many cases that the published algorithms are not quite perfect and had to be corrected and revised. Those algorithms taken from the literature, plus some new ones which had to be written, were all checked out on the UNIVAC 1107 thin-film memory computer using 1107 ALGOL, an extension of ALGOL 60 (Programmers' Guide, 1965). Certain problems which arose because of the limited range ($10^{\pm 38}$) of floating-point numbers on the 1107 were checked with a FORTRAN program on the IBM 1620, which has a somewhat more extended programmed floating-point range ($10^{\pm 99}$). In certain regions of $F_n^{(-)}$ and $G_n^{(-)}$ even the range of 1620 programmed floating point was not adequate to prevent characteristic overflow and underflow. It is expected that a recomputation of the functions on the UNIVAC 1108, which has a floating point of $10^{\pm 300}$ will allow computation of values of $F_n^{(-)}$ and $G_n^{(-)}$ in all regions of practical interest without problems of characteristic underflow or overflow. Computations on the 1108 will also provide a further numerical check on accuracy, since the Jacobi series for $F_n^{(-)}$ and $G_n^{(-)}$ can be evaluated for 25 or 30 terms rather than just 16.

The primary difficulty in calculating $F_n^{(-)}$ and $G_n^{(-)}$ with this general algorithm, once the specific algorithms for the various mathematical functions were corrected, was found in computing the imaginary parts of the functions. The imaginary parts of the near-zone and source-region functions contain a multiplied Neumann function whose value goes toward minus infinity near zero, and does so more rapidly as the order of the function increases. Each term of the series for $Im(F_n^{(-)})$ or $Im(G_n^{(-)})$ was within the floating-point range of the computer; however, each is the product of a function tending toward minus infinity (Neumann) and a function tending toward plus infinity (Associated Legendre) as the summation index of the series increases. This overflow problem was worse in $G_n^{(-)}$ than in $F_n^{(-)}$ since the arguments r/a of $G_n^{(-)}$ are much smaller, being always less than unity in the source region. In addition, the algorithm for $P_n^m(x)$ caused floating-point overflow on the 1107 for m greater than 16. This value set the upper limit of summation. Since the function orders on the second form of the Jacobi series (20) and (21) are smaller than those of the first form (10) and (12), it was thought that the second series would be more efficient for computation. The second form of the series was programmed in 1107 ALGOL and the first form in 1620 FORTRAN. The results compared to six decimal places (with an upper summation limit of 16) for all values of $F_n^{(-)}$ and $G_n^{(-)}$ for which floating-point overflow did not occur. In the latter case, there was a degradation in the comparison since the higher-order terms of each series caused overflow. In some such cases, only three figure compared and in a few bad cases only one or two; however, the over-all comparison was encouraging since completely different series forms, approximations, algorithms, programming languages, computers and arithmetic were used.

4.2 *The general algorithm*

The algorithm FN and GN for $F_n^{(-)}$ and $G_n^{(-)}$, respectively, are given with input parameters $N =$ order, $BA = \beta{*}a$, $RA = r/a$, $THETA = \theta$ and output values $RGN = Re(GN)$ and $IGN = Im(GN)$ for $G_n^{(-)}$, and $RFN = Re(FN)$ and $IFN = Im(FN)$ for $F_n^{(-)}$. These procedures call upon the real procedure $COEFF$ to calculate

$$\frac{(2m+2n+1)m!}{(m+2n)!}\,P_{m+n}^{n}(\cos\theta);$$

the real procedure *SPHBES* to calculate spherical Bessel functions; and the real procedure *SPHBEN* to calculate spherical Neumann functions (Herndon 1961). The latter procedure allows the computation of the Hankel function as:

$$h_n^{(2)} = j_n - iy_n.$$

Although the values of $F_n^{(-)}$ and $G_n^{(-)}$ could have been calculated in complex arithmetic in 1107 ALGOL, it was simpler to do two parallel computations since the imaginary part of $h_n^{(2)}$ is the only imaginary number in the computation. If these algorithms are implemented as complex procedures for any other ALGOL implementation with complex arithmetic or as complex functions in FORTRAN IV, it might shorten the time of computation to deal directly with complex numbers.

LEGENDREA is a real procedure that calculates the associated Legendre function $P_n^m(x)$. It is a somewhat modified version of ACM 47 (Herndon, 1961) to allow calculation of $P_n^m(0)$. *GAMMA*, a real procedure for the Gamma function ACM 34 (Lipp, 1961), is used for computing $P_n^m(0)$ by

$$P_n^m(0) = \frac{(-1)^{m+n}\Gamma\left(\frac{m+n+1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{n-m+1}{2}\right)}$$

inside the general algorithm for $P_n^m(x)$ (Erdelyi, 1955c).

SPHBEN, *LEGENDREA* and *GAMMA* can be found in the cited literature; the other procedures are given following:

```
procedure FN (N, BA, RA, THETA, RFN, IFN); value
  N, BA, RA, THETA;
integer N; real BA, RA, THETA, RFN, IFN;
comment Computes the near-zone field FN = RFN+iIFN
given the order N and the coordinates BA = B × A,
RA = R/A and THETA. This procedure uses
LEGENDREA Associated Legendre function, SPHBES
spherical Bessel function, SPHBEN spherical Neumann
function and COEFF, a special coefficient;
begin integer M; real ANGLE;
ANGLE := cos (3·1415926 × THETA/180·0);
RFN := IFN := 0·0;
for M := 0 step 1 until 16 do
begin
RFN := RFN + COEFF(M,N)×SPHBES(M+N,BA)
    ×SPHBES(M+N,BA×RA)×LEGENDREA
    (N,M+N,ANGLE,0·0);
IFN := IFN + COEFF(M,N)×SPHBES(M+N,BA)
    ×SPHBEN(M+N,BA×RA)×
    LEGENDREA(N,M+N,ANGLE,0·0);
end; RFN :=((−1) ↑ N)×RFN; IFN :=((−1) ↑ N)
    ×IFN
end FN;


procedure GN (N, BA, RA, THETA, RGN, IGN); value
    N, BA, RA, THETA;
integer N; real BA, RA, THETA, RGN, IGN;
comment Computes the source-region function
GN = RGN + iIGN given the order N and the co-
    ordinates BA = B×A,
RA = R/A and THETA;
begin integer M; real ANGLE;
ANGLE := cos (3·1415926×THETA/180·0);
RFN := IFN := 0·0;
for M := 0 step 1 until 16 do
begin
RGN := RGN+COEFF(M,N)×SPHBES(M+N,
    BA×RA)×SPHBES(M+N,BA)×
    LEGENDREA(N,M+N,ANGLE,0·0);
IGN := IGN+COEFF(M,N)×SPHBES(M+N,
    BA×RA)×SPHBEN(M+N,BA)×
    LEGENDREA(N,M+N,ANGLE,0·0)
end; RGN := ((− 1) ↑ N)×RGN; IGN := ((−1) ↑ N)
    ×IGN
end GN;
real procedure COEFF (M,N); value M,N; integer M,N;
comment Computes special series coefficient for FN and
    GN;
begin real FACM, FACMN; integer K;
FACM := FACMN := 1·0;
for K :=1 step 1 until M do FACM := FACM ×
    (FACM + 1·0);
for K := 1 step 1 until M + 2×N do FACMN :=
    FACMN×(FACMN+1·0);
COEFF := (2×M+2×N+1)×FACM×
    LEGENDREA(N,M+N,0·0,0·0)/FACMN
end COEFF;

real procedure SPHBES (N,X); value N,X; integer N;
    real X;
comment Computes spherical Bessel function by a tech-
nique similar to ACM algorithm 49, with a correction due
to M. Dacic of the University of Paris;
begin
real array BJ [1 : 20]; integer M, L, I, L1, L2;
real BETA, BES, Y, FAC1;
M := N+1; L := 2×N+1; FAC1 := 1·0;
if X = 0 then begin
    if N = 0 then
        begin BES := 0; go to GATE end
    else
    begin BES := 1·0; go to GATE end end;
if N> 16 go to GATE; if N> 10 go to NEXT;
if (X−0·6−0·4×N) > 0 then go to COMP;
go to FIX;
NEXT: if (X−4·6−0·2×N) > 0 then go to COMP;
FIX: for I := 1 step 2 until L do FAC1 :=FAC1×I;
    BETA := (1·0−(X×X/(2·0×(2×N + 3·0)))
        +(X ↑ 4/(8·0×(2× N+3·0)×
        (2×N+5·0)))−(X ↑ 6/(48·0×(2×N + 3·0)
        ×(2×N+5·0)×(2×N+7·0))))/FAC1;
BES := 1·0; for I := 1 step 1 until N do BES :=BES×X;
BES := BES×BETA; go to GATE;
COMP: BJ[1] :=sin(X)/X;
    BJ [2] := (BJ [1] − cos(X))/X;
    for I := 3 step 1 until M do
    begin L1 := I − 1; L2 := I − 2;
        Y := 2×I − 3;
        BJ [I] := (Y×BJ [L1] /X)−BJ [L2]
end;
BES := BJ [M];
GATE: SPHBES := BES
end SPHBES;
```

4.3 *Computational results*

A 'driver' or main program was written to 'drive' *FN* and *GN* through a sequence of computations to generate a table of values of $F_n^{(-)}$ with angle $\theta = 0$, 10, 20, 30, 40, 50, 60, 80, 90 degrees, ratios r/a equal to 1·1, 1·25, 1·5, 2, 3, 5 for orders $N = 0$, 1, 2, 4 and values of loop radius equal to 0·2, 0·5, 1·0, 2·0, 5·0. $G_n^{(-)}$ was tabulated for the same arguments with the exception that its r/a values were set to 0·05, 0·1, 0·25, 0·5, 0·75, 0·9. The time required to calculate these 2400 eighteen-digit complex values of $F_n^{(-)}$ and $G_n^{(-)}$ on the 1107 was

ORDER N=0 SOURCE RADIUS A= .50/BETA

THETA DEGREES		.F SUB N (A,THETA,R) FOR VALUES OF R/A EQUAL TO 1.10	1.25	1.50	2.00	3.00	5.00
.0	REAL	9.111847,-01	8.975695,-01	8.713618,-01	8.066876,-01	6.373087,-01	2.288748,-01
	IMAG	-1.346166, 00	-1.157497, 00	-8.903117,-01	-5.017003,-01	-4.063617,-02	3.085577,-01
10.0	REAL	9.110116,-01	8.973472,-01	8.710458,-01	8.061432,-01	6.361913,-01	2.265942,-01
	IMAG	-1.281960, 00	-1.095664, 00	-8.447244,-01	-4.762399,-01	-2.975363,-02	3.124393,-01
20.0	REAL	9.108897,-01	8.971908,-01	8.708233,-01	8.057601,-01	6.354050,-01	2.249898,-01
	IMAG	-1.226426, 00	-1.056816, 00	-8.165192,-01	-4.598174,-01	-2.239524,-02	3.151324,-01
30.0	REAL	9.108338,-01	8.971191,-01	8.707213,-01	8.055844,-01	6.350445,-01	2.242543,-01
	IMAG	-1.216635, 00	-1.041730, 00	-8.045238,-01	-4.526468,-01	-1.909832,-02	3.163569,-01
40.0	REAL	9.116388,-01	8.981521,-01	8.721905,-01	8.081150,-01	6.402393,-01	2.348628,-01
	IMAG	-2.324479, 00	-1.689647, 00	-1.152268, 00	-6.031457,-01	-7.421537,-02	2.978319,-01
50.0	REAL	9.116388,-01	8.981521,-01	8.721905,-01	8.081150,-01	6.402393,-01	2.348628,-01
	IMAG	-2.324479, 00	-1.689647, 00	-1.152268, 00	-6.031457,-01	-7.421537,-02	2.978319,-01
60.0	REAL	9.116388,-01	8.981521,-01	8.721905,-01	8.081150,-01	6.402393,-01	2.348628,-01
	IMAG	-2.324479, 00	-1.689647, 00	-1.152268, 00	-6.031457,-01	-7.421537,-02	2.978319,-01
70.0	REAL	9.116388,-01	8.981521,-01	8.721905,-01	8.081150,-01	6.402393,-01	2.348628,-01
	IMAG	-2.324479, 00	-1.689647, 00	-1.152268, 00	-6.031457,-01	-7.421537,-02	2.978319,-01
80.0	REAL	9.116388,-01	8.981521,-01	8.721905,-01	8.081150,-01	6.402393,-01	2.348628,-01
	IMAG	-2.324479, 00	-1.689647, 00	-1.152268, 00	-6.031457,-01	-7.421537,-02	2.978319,-01
90.0	REAL	9.116388,-01	8.981521,-01	8.721905,-01	8.081150,-01	6.402393,-01	2.348628,-01
	IMAG	-2.324479, 00	-1.689647, 00	-1.152268, 00	-6.031457,-01	-7.421537,-02	2.978319,-01

1 hour 13 minutes and 40 seconds. These tables and the check results computed on the 1620 are far too extensive to present here, but an example of the results might be of interest. The Table opposite is a typical page of output from the table builder program.

5. Conclusions

The foregoing algorithms for $F_n^{(-)}$ and $G_n^{(-)}$ are easily adapted to the numerical evaluation of the respective complex conjugate quantities $F_n^{(+)}$ and $G_n^{(+)}$. The user of these algorithms for $F_n^{(\pm)}$ and $G_n^{(\pm)}$ may find it convenient to prepare a set of tables on his computer by writing and running a driver program which calculates the values most useful to him and formats them in a manner convenient for his application. If the $F_n^{(\pm)}$ and $G_n^{(\pm)}$ values are only part of a more extensive computation, it would, of course, be more useful to imbed these procedures within the larger program. If the potential user's computer does not have an ALGOL compiler (an unlikely possibility these days), it is simple enough to hand translate the procedures given in this paper into FORTRAN IV subroutines and functions.

References

Association for Computing Machinery (1960). "Algorithms," *Comm. ACM*, Vol. 3, p. 73.

British Computer Society (1964). "Algorithm Supplement," *The Computer Bulletin*, Vol. 8, No. 2.

ERDELYI, A. (editor) (1955a). *Higher Transdental Functions*, McGraw-Hill Book Company, Inc., New York, Vol. 2, p. 102.

ERDELYI, A., op cit., Vol. 2, p. 183.

ERDELYI, A., op. cit., Vol. 1, p. 145.

FOSTER, D. (1944). "Loop Antennas with Uniform Current," *Proc. IRE*, Vol. 32, p. 603.

GALEJS, J., and THOMPSON, T. W. (1962). "Admittance of a Cavity-Backed Annular Slot Antenna," *IRE Trans.*, Vol. AP-10, p. 671.

HERNDON, J. R. (1961). Algorithm 47: "Associated Legendre Function of the First Kind," and Algorithm 49: "Spherical Neumann Function," *Comm. ACM*, Vol. 4, p. 178.

HU, M. K. (1960). "Fresnel Region Field Distributions of Circular Aperture Antennas," *IRE Trans.*, Vol. AP-8, p. 344.

KING, R. W. P. (1964). "Theory of the Terminated Insulated Antenna in a Conducting Medium," *IEEE Trans.*, Vol. AP-12, p. 305.

LAVINE, H., and PAPUS, C. H. (1951). "Theory of the Circular Diffraction Antenna," *Jour. Appl. Phys.*, Vol. 22, p. 29.

LIPP, M. F. (1961). Algorithm 34: "Gamma Function," *Comm. ACM*, Vol. 4, p. 106.

MARTIN, E. J., Jr. (1960). "Radiation Fields of Circular Loop Antennas by a Direct Integration Process," *IRE Trans.*, Vol. AP-8, p. 105; "Correction," Vol. AP-8, p. 515.

MARTIN, E. J., Jr. (1963). "Exact Expressions for the Vector Potential Produced by Circular Loop Antennas," *Proc. IEEE*, Vol. 51, p. 1042.

MARTIN, E. J., Jr. (1964). "Far-Zone, Near-Zone, and Antenna-Region Fields of the Circular Loop Antenna," Ph.D. Thesis, University of Kansas, Lawrence, Kansas.

NAUR, P. (editor) (1960). Report on the algorithmic language ALGOL 60, *Comm. ACM*, Vol. 3, p. 299.

RAMO, S., and WHINNERY, J. R. (1947). *Fields and Waves in Modern Radio*, John Wiley and Sons, Inc., New York, N.Y., p. 448.

SESHARDI, S. R., and WU, T. T. (1960). "High-Frequency Diffraction of Electromagnetic Waves by a Circular Aperture in an Infinite Conducting Screen," *IRE Trans.*, Vol. AP-8, p. 27.

SESHARDI, S. R., and WU, T. T. (1963). "Diffraction by a Circular Aperture in a Unidirectionally Conducting Screen," *IEEE Trans.*, Vol. AP-11, p. 56.

SILVER, S. (1939). *Microwave Antenna Theory and Design*, M.I.T. Rad. Lab. Series, Vol. 12, McGraw-Hill Book Company, Inc., New York, N.Y., p. 169.

STRATTON, J. A. (1941). *Electromagnetic Theory*, McGraw-Hill Book Company, Inc., New York, N.Y., p. 404.

SZEGO, G. (1959). "Orthogonal Polynomials," American Mathematical Society, New York, N.Y., p. 243.

WAIT, J. R. (1959). "Electromagnetic Radiation from Cylindrical Structures," Pergamon Press, New York, N.Y., p. 182.

WAIT, J. R., op. cit., p. 193.

UNIVAC Division of Sperry Rand Corp. (1965). *UNIVAC* 1107 *ALGOL Programmers' Guide*, New York.