

# The calculation of Lamé polynomials

By J. H. Wilkinson\*

The coefficients of Lamé polynomials may be determined from the eigenvectors of certain tri-diagonal matrices. Although these matrices have extraordinarily ill-conditioned eigensystems it is shown that there is a simple procedure by means of which the eigenvectors may be determined accurately without resorting to high-precision arithmetic.

## 1. Introduction

In a recent publication Arscott and Khabaza (1962) discussed the calculation of Lamé polynomials and gave extensive tables of the coefficients. The accurate determination of these coefficients was based on a minor modification of a method suggested by Wilkinson; from the point of view of numerical stability this method has rather remarkable properties and these form the subject of this paper.

The Lamé polynomials arise in connexion with the solution of Lamé's equation

$$\frac{d^2 w}{dz^2} + \{\lambda - n(n+1)k^2 \operatorname{sn}^2 z\} w = 0 \quad (1)$$

where  $\operatorname{sn} z \equiv \operatorname{sn}(z, k)$  is the Jacobian elliptic function. When  $n$  is an integer (which we may take to be positive since the equation (1) is invariant with respect to the substitution  $n \rightarrow -n-1$ ) and  $\lambda$  takes one of a set of  $2n+1$  eigenvalues, the equation has solutions of the form

$$w = \operatorname{sn}^\sigma z \operatorname{cn}^\sigma z \operatorname{dn}^\tau z F(\operatorname{sn}^2 z), \quad (2)$$

where  $\rho, \sigma, \tau = 0$  or  $1$ , and  $F(\operatorname{sn}^2 z)$  is a polynomial of degree  $\frac{1}{2}(n - \rho - \sigma - \tau)$ . There are thus eight types of solution corresponding to the eight possible combinations of  $\rho, \sigma, \tau$ . Since all eight types behave in much the same way as regards the phenomenon we wish to discuss, we concentrate on type 1 for which  $\rho = \sigma = \tau = 0$ . This type exists only when  $n$  is even ( $n = 2N$ ) and there are then  $N+1$  such polynomials. If we write

$$F(\operatorname{sn}^2 z) = \sum_{r=0}^N (-1)^r x_r \operatorname{sn}^{2r} z \quad (3)$$

then it can be shown that the  $x_r$  satisfy relations of the form

$$a_r x_{r-1} + (b_r - \lambda) x_r + c_r x_{r+1} = 0 \quad (r = 0, \dots, N) \quad (4)$$

with  $x_{-1} = x_{N+1} = 0$ . Hence each  $\lambda$  is an eigenvalue of the matrix  $A$  given by

$$A = \begin{bmatrix} b_0 & c_0 & & & \\ a_1 & b_1 & c_1 & & \\ & a_2 & b_2 & c_2 & \\ & & \ddots & \ddots & \ddots \\ & & & a_N & b_N \end{bmatrix} \quad (5)$$

and the  $x_r$  are proportional to the elements of the corresponding eigenvector. For the polynomial of type 1 the elements of  $A$  are given by

$$a_r = -(2N - 2r + 2)(2N + 2r - 1)k^2, \\ b_r = 4r^2(1 + k^2), \quad c_r = (2r + 2)(2r + 1). \quad (6)$$

Since  $a_r c_{r-1}$  is negative the eigenvalues of  $A$  cannot be found by the Sturm sequence method, Givens (1953). However, if we write

$$F(\operatorname{sn}^2 z) = \sum_{r=0}^N y_r \operatorname{cn}^{2r} z, \quad (7)$$

it can be shown that the eigenvalues  $\lambda$  of  $A$  are also those of the matrix  $B$  defined by

$$B = \begin{bmatrix} b'_0 & c'_0 & & & \\ a'_1 & b'_1 & c'_1 & & \\ & a'_2 & b'_2 & c'_2 & \\ & & \ddots & \ddots & \ddots \\ & & & a'_N & b'_N \end{bmatrix}, \quad (8)$$

where

$$a'_r = -(2N - 2r + 2)(2N + 2r - 1)k^2, \\ b'_r = 2N(2N + 1)k^2 - 4r^2(k^2 - k'^2), \\ c'_r = -(2r + 2)(2r + 1)k'^2, \quad k'^2 = 1 - k^2. \quad (9)$$

Since  $a'_r c'_{r-1}$  is positive the eigenvalues of  $B$  can be found using the Sturm sequence property, and it is well known that this method gives very accurate results.

## 2. Ill-condition of $A$ matrices

To find the  $x_r$  we have only to find the eigenvectors of  $A$  using the accurate eigenvalues determined from  $B$ . In normal circumstances the method of inverse iteration (Wielandt, 1944) is well suited to such a requirement, but unfortunately for most of the relevant values of the parameters the matrices of type  $A$  have some very ill-conditioned eigenvalues and eigenvectors.

The severity of the ill-condition of the eigenvalues is well-illustrated in Table 1. We give first the elements  $a_r, b_r, c_r$  of the matrix  $A$  corresponding to the values  $N = 12, k^2 = 0.9$  followed by accurate eigenvalues  $\lambda_i (i = 1, \dots, 13)$ . We then give accurate eigenvalues of a matrix  $\tilde{A}$  which is equal to  $A$  except that the element  $b_{10}$ , which is 760, is replaced by  $760 + 10^{-1} \cdot 2^{-16}$ .

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Table 1

$a_r$	$b_r$	$c_r$	EIGENVALUES OF $A$	EIGENVALUES OF $\bar{A}$
	0.0	2.0	22.76771 22	22.76771 22
-540.0	7.6	12.0	110.03760 3	110.03760 3
-534.6	30.4	30.0	189.70299 1	189.70298 6
-522.0	68.4	56.0	261.75802 7	261.75824 8
-502.2	121.6	90.0	326.19293 8	326.18811 7
-475.2	190.0	132.0	382.99035 4	383.05092 3
-441.0	273.6	182.0	432.11679 8	431.65139 7
-399.6	372.4	240.0	473.50004 0	476.33090 5
-351.0	486.4	306.0	506.94812 2	499.61841 6
-295.2	615.6	380.0	531.25251 2	538.87962 6
-232.2	760.0	462.0	545.02985 6	$\pm 12.55463 25i$
-162.0	919.6	552.0	565.16826 7	569.09605 8
-84.6	1094.4		592.53478 0	592.03838 2

$\bar{A} = A + 10^{-1} 2^{-16} E_{11, 11}$

(The matrix was read in with a scale factor of 10 to avoid rounding errors; the change is therefore 1 in the least-significant digit of the 30-digit mantissa used on DEUCE.) It will be seen that some of the eigenvalues of  $\bar{A}$  differ very substantially from those of  $A$ . That the corresponding eigenvectors are also sensitive to small changes in the elements  $a_r$ ,  $b_r$ ,  $c_r$  is evident from the relations (4).

Now error analysis of inverse iteration shows that working in floating-point with  $t$  digits in the mantissa each computed eigenvector  $x^{(i)}$  of  $A$  is an exact eigenvector of  $A + E_i$ , where  $\|E_i\|_2$  is of order  $n^{1/2-t}\|A\|_2$  (Wilkinson, 1963). In view of this it is scarcely to be expected that single-precision arithmetic will give eigenvectors of acceptable accuracy. Further, the results in Table 1 show that if rounding errors are made in converting the elements  $a_r$ ,  $b_r$ ,  $c_r$  to the binary scale (as indeed is inevitable unless a scale factor is incorporated in  $A$ , since some of the  $a_r$ ,  $b_r$  and  $c_r$  are non-terminating binary numbers for required values of  $k$ ) then the matrix  $\bar{A}$  stored in the computer may well have very different eigenvalues from those of  $A$ , in which case relations (4) show that the eigenvectors will also differ substantially.

From now on we shall use the bar in a rather general way to denote an approximate quantity, not necessarily the same one each time. Each use of the bar will be accompanied by a local definition of its precise meaning.

### 3. Calculation of accurate eigenvectors of $A$

Normally when a matrix is as ill-conditioned as this, high-precision arithmetic is essential for the determination of an accurate eigensystem. In this case, however, the form of the elements  $a_r$ ,  $b_r$  and  $c_r$  makes it possible to calculate accurate eigenvectors in a very economical manner. A remarkable feature of the method is that accurate eigenvectors of  $A$  can be found even using an  $\bar{A}$  of the type discussed at the end of

Section 2, in spite of the fact that  $\bar{A}$  does not possess the eigenvectors we wish to determine!

Before describing the method we examine some of the properties of  $A$  and its eigensystem. First the eigenvalues of  $A$  are real since they are also eigenvalues of the quasi-symmetric matrix  $B$ . The quantities  $a_i$ ,  $b_i$ ,  $c_i$  are such that

$$a_i < 0, b_i > 0, c_i > 0, \quad (10)$$

and from these conditions and the monotonicity of the  $b_i$  it can be shown that

$$b_0 < \lambda < b_N. \quad (11)$$

Now if  $\lambda$  is an exact eigenvalue the corresponding  $x_i$  satisfy the equations

$$\left. \begin{aligned} (b_0 - \lambda)x_0 + c_0x_1 &= 0 \\ a_ix_{i-1} + (b_i - \lambda)x_i + c_ix_{i+1} &= 0 \quad (i=1, \dots, N-1) \\ a_Nx_{N-1} + (b_N - \lambda)x_N &= 0 \end{aligned} \right\} \quad (12)$$

Only  $N$  of these equations are needed to determine the ratios of the  $x_i$ ; the  $x_i$  thus determined automatically satisfy the remaining equation. In the numerical calculation we propose to omit the equation containing the  $b_s$  for which  $|b_s - \lambda|$  takes its minimum value. Since the  $b$  are monotonic increasing we have

$$b_i - \lambda < 0 \quad (i < s), \quad b_i - \lambda > 0 \quad (i > s). \quad (13)$$

We take  $x_0 = 1$  and determine  $x_1, x_2, \dots, x_s$  from equations 1 to  $s$  of the set (12) and then take  $x'_N = 1$  and determine  $x'_{N-1}, x'_{N-2}, \dots, x'_s$  from equations  $N+1$  to  $s+2$ . We have

$$\left. \begin{aligned} x_1 &= -(b_0 - \lambda)/c_0 \\ x_{i+1} &= \{-a_ix_{i-1} - (b_i - \lambda)x_i\}/c_i \\ &\quad i = 1, \dots, s-1 \\ x'_{N-1} &= (b_N - \lambda)/(-a_N) \\ x'_{i-1} &= (b_i - \lambda)x'_i + c_ix'_{i+1}/(-a_i) \\ &\quad i = N-1, \dots, s+1 \end{aligned} \right\} \quad (14)$$

From relations (10) and (13) it is obvious that all  $x_i$  and  $x'_i$  are positive. The sequence

$$x_0, x_1, \dots, x_s, kx'_{s+1}, \dots, kx'_N \quad (k = x_s/x'_s) \quad (15)$$

gives the required eigenvector of  $A$ . It can subsequently be normalized if this is required.

Obviously if  $\lambda$  is exact and the computation is performed exactly we obtain an exact eigenvector of  $A$ . We assert that if  $\bar{\lambda}$  is a correctly rounded eigenvalue of  $A$  and the above process is performed in floating-point arithmetic using  $\bar{\lambda}$ , then the computed vector will be an accurate eigenvector of  $A$ . *This will be true even if we use rounded values  $\bar{a}_i, \bar{b}_i, \bar{c}_i$ , in spite of the fact that the eigenvectors of  $\bar{A}$  are not close to those of  $A$ . The only requirement is that  $\bar{\lambda}$  should be a correctly rounded eigenvalue of  $A$  itself and not of  $\bar{A}$ . Since  $\bar{\lambda}$  can be found from  $B$  which is well-conditioned this requirement presents no difficulty.*

#### 4. "Stability" of determination of the eigenvectors

We now consider the computation of an eigenvector using floating-point arithmetic with a  $t$ -digit mantissa. The error bounds we shall obtain are not very sensitive to the particular rounding procedure that is used. For definiteness we shall assume that if  $a$  and  $b$  are standard floating-point numbers then

$$\left. \begin{aligned} \text{fl}(a \pm b) &= a(1 + \varepsilon_1) \pm b(1 + \varepsilon_2) \\ \text{fl}(a \times b) &= ab(1 + \varepsilon_3) \quad (|\varepsilon_i| \leq 2^{-t}), \\ \text{fl}(a \div b) &= a(1 + \varepsilon_4)/b \end{aligned} \right\} \quad (16)$$

where  $\text{fl}(a + b)$ , for example, means the result of adding  $a$  and  $b$  using floating-point arithmetic (see, for example, Wilkinson, 1963).

We shall describe  $\bar{A}$  as being a neighbouring matrix of the tri-diagonal matrix  $A$  if we have

$$\left. \begin{aligned} \bar{a}_i &= a_i(1 + p_i) \\ \bar{b}_i &= b_i(1 + q_i) \quad (|p_i|, |q_i|, |r_i| \leq 2^{-t}) \\ \bar{c}_i &= c_i(1 + r_i) \end{aligned} \right\} \quad (17)$$

We let  $x_i$  and  $x'_i$  denote the exact quantities corresponding to equations (14) using an exact eigenvalue  $\lambda$  of  $A$ , while  $\bar{x}_i$  and  $\bar{x}'_i$  denote the computed values obtained using floating-point arithmetic and  $\bar{a}_i, \bar{b}_i, \bar{c}_i$  and  $\bar{\lambda}$ . For definiteness we assume that  $\bar{a}_i, \bar{b}_i, \bar{c}_i, \bar{\lambda}$  are correctly rounded values so that  $\bar{a}_i, \bar{b}_i, \bar{c}_i$  satisfy relations (17) and

$$\bar{\lambda} = \lambda(1 + s_i) \quad (|s_i| \leq 2^{-t}). \quad (18)$$

Our object is to show that  $\bar{x}_i/x_i$  and  $\bar{x}'_i/x'_i$  are both very close to unity, and therefore that the computed vector is very close to the true eigenvector of  $A$ . We write

$$\bar{x}_i/x_i = 1 + \alpha_i, \quad \bar{x}'_i/x'_i = 1 + \beta_i \quad (19)$$

and we require bounds for  $\alpha_i$  and  $\beta_i$ . For  $\bar{x}_{i+1}$  we have

$$\bar{x}_{i+1} = \text{fl}[-\bar{a}_i\bar{x}_{i-1} - (\bar{b}_i - \bar{\lambda})\bar{x}_i/\bar{c}_i], \quad (20)$$

where this notation implies that the expression in square brackets is computed using floating-point arithmetic with the previously computed values of  $\bar{x}_{i-1}$  and  $\bar{x}_i$ .

Now in the numerator of (20) both  $-\bar{a}_i\bar{x}_{i-1}$  and  $-(\bar{b}_i - \bar{\lambda})\bar{x}_i$  are positive from our definition of  $s$ , so that no cancellation takes place when the quantity in square brackets is computed. We write

$$\text{fl}(\bar{b}_i - \bar{\lambda}) = (b_i - \lambda)(1 + \gamma_i) \quad (21)$$

and obtain bounds for  $\gamma_i$  later. Repeated application of relations (16) then shows that

$$\bar{x}_{i+1} = \frac{-a_i x_{i-1}(1 + v_i) - (b_i - \lambda)x_i(1 + w_i)}{c_i}, \quad (22)$$

where

$$\begin{aligned} (1 + \alpha_{i-1})(1 - 2^{-t})^5 &\leq (1 + v_i) \\ &\leq (1 + \alpha_{i-1})(1 + 2^{-t})^5, \end{aligned} \quad (23)$$

$$\begin{aligned} (1 + \alpha_i)(1 + \gamma_i)(1 - 2^{-t})^4 &\leq (1 + w_i) \\ &\leq (1 + \alpha_i)(1 + \gamma_i)(1 + 2^{-t})^4. \end{aligned} \quad (24)$$

Since we have

$$x_{i+1} = \frac{-a_i x_{i-1} - (b_i - \lambda)x_i}{c_i} \quad (25)$$

and  $-a_i x_{i-1}$  and  $-(b_i - \lambda)x_i$  are positive, equations (22) and (25) imply that  $\alpha_{i+1}$  lies between  $v_i$  and  $w_i$ .

In order to obtain bounds for  $\alpha_{i+1}$  we require a bound for  $\gamma_i$ . We have

$$\begin{aligned} \text{fl}(\bar{b}_i - \bar{\lambda}) &= \bar{b}_i(1 + \varepsilon_1) - \bar{\lambda}(1 + \varepsilon_2)(|\varepsilon_1|, |\varepsilon_2| \leq 2^{-t}) \\ &= b_i(1 + \eta_1) - \lambda(1 + \eta_2), \end{aligned} \quad (26)$$

$$\text{where } (1 - 2^{-t})^2 \leq 1 + \eta_k \leq (1 + 2^{-t})^2 \quad (k = 1, 2). \quad (27)$$

Hence

$$(1 + \gamma_i) = 1 + \frac{b_i \eta_1 - \lambda \eta_2}{b_i - \lambda}, \quad (28)$$

giving

$$|\gamma_i| \leq \frac{\lambda + b_i}{-b_i \lambda} (2 \cdot 2^{-t} + 2^{-2t}). \quad (29)$$

Our method of choosing  $s$  ensures that

$$\lambda > \frac{1}{2}(b_{s-1} + b_s).$$

From this it follows that

$$\frac{\lambda + b_i}{\lambda - b_i} < \frac{2s^2 - 2s + 1 + 2_i^2}{2s^2 - 2s + 1 - 2_i^2} = f(s, i). \quad (30)$$

since  $(\lambda + b_i)/(\lambda - b_i)$  decreases as  $\lambda$  increases.

Hence we have

$$2 - (1 + 2^{-t})^{2f(s, i)} < (1 + \gamma_i) < (1 + 2^{-t})^{2f(s, i)}. \quad (31)$$

From (23), (24) and (31) it is easy to show by induction that

$$2 - (1 + 2^{-t})^{g(s, i)} < \bar{x}_i/x_i < (1 + 2^{-t})^{g(s, i)} \quad (32)$$

where  $g(s, i) = 5(i - 1) + 2 \sum_{j=0}^{i-1} f(s, j)$ .

Table 2

$N = 12$ ,  $k^2 = 0.9$ ,  $\bar{\lambda} = 592.534780$ . Calculation of eigenvector using nine significant decimals

FORWARD SEQUENCE	BACKWARD SEQUENCE	NORMALIZED EIGENVECTOR
$\bar{x}_0 = (1.00000\ 000)10^0$	$\bar{x}'_{12} = (1.00000\ 000)10^0$	$(4.19238\ 624)10^{-9}$
$\bar{x}_1 = (2.96267\ 390)10^2$	$\bar{x}'_{11} = (5.93221\ 300)10^0$	$(1.24206\ 733)10^{-6}$
$\bar{x}_2 = (1.44864\ 250)10^4$	$\bar{x}'_{10} = (1.53840\ 775)10^1$	$(6.07326\ 888)10^{-5}$
$\bar{x}_3 = (2.76723\ 593)10^5$	$\bar{x}'_9 = (2.28982\ 788)10^1$	$(1.16013\ 218)10^{-3}$
$\bar{x}_4 = (2.72504\ 238)10^6$		$(1.14244\ 302)10^{-2}$
$\bar{x}_5 = (1.58031\ 980)10^7$		$(6.62531\ 102)10^{-2}$
$\bar{x}_6 = (5.80020\ 983)10^7$		$(2.43167\ 199)10^{-1}$
$\bar{x}_7 = (1.39934\ 598)10^8$		$(5.86659\ 883)10^{-1}$
$\bar{x}_8 = (2.24925\ 460)10^8$		$(9.42974\ 404)10^{-1}$
$\bar{x}_9 = (2.38527\ 641)10^8$		$(1.00000\ 000)10^0$
		$(6.71844\ 275)10^{-1}$
		$(2.59068\ 075)10^{-1}$
		$(4.36714\ 045)10^{-2}$
		$(4.36714\ 045)10^{-2}$

The tables computed by Arscott and Khabaza covered values of  $N$  up to 30 and obviously the maximum value of  $g(s, i)$  is attained where  $s = N$  and  $i = N - 1$ . It is easy to show that

$$\sum_{i=0}^{s-1} f(s, i) \sim s \log 4s \quad (33)$$

and we certainly have the overall bound

$$2 - (1 + 2^{-t})^{400} < \bar{x}_i/x_i < (1 + 2^{-t})^{400}. \quad (34)$$

On a computer with  $t = 30$  we can therefore be certain that  $\bar{x}_i$  agrees with  $x_i$  to at least six significant figures, the accuracy required for the tables. The analysis we have just given can be sharpened considerably, but only at the expense of rather undesirable complexity. In practice the statistical distribution of errors alone is likely to ensure that  $\bar{x}_i/x_i$  satisfies some such relation as

$$2 - (1 + 2^{-t})^{20} < \bar{x}_i/x_i < (1 + 2^{-t})^{20}. \quad (35)$$

Similar arguments apply to the  $\bar{x}'_i$ .

An interesting feature of this result is that it shows that if some elements of  $x_i$  are much smaller than others, the former will have much smaller *absolute* errors since our method ensures small *relative* errors.

In Table 2 we give the computed vector of the matrix  $A$  with  $N = 12$ ,  $k^2 = 0.9$  corresponding to  $\bar{\lambda} = 592.534780$ , which is a correctly rounded eigenvalue. The corresponding value of  $s$  is nine so that  $\bar{x}_0, \dots, \bar{x}_9$  are determined from the first nine equations and  $\bar{x}_{12}, \dots, \bar{x}_9$  from the last three. The computation was performed retaining nine significant figures at each stage. The maximum error in any component of the normalized eigenvector is 3 in the ninth figure, so that the error is well below the upper bound given above.

## 5. Eigenvectors of neighbouring matrices

The result of the previous section shows that we can obtain an accurate eigenvector of  $A$  while using an  $\bar{A}$  for the calculations. By exactly the same argument we can show that if  $\bar{A}$  is any other neighbouring matrix we can find its eigenvectors by using  $\bar{A}$  provided the value  $\bar{\lambda}$  which we use is a close approximation to an eigenvalue of  $\bar{A}$ . From this we see immediately that if  $A_1$  and  $A_2$  are two neighbouring matrices of  $A$  and if  $A_1$  has an eigenvalue which is very close to an eigenvalue of  $A_2$ , then the corresponding eigenvectors of  $A_1$  and  $A_2$  are also very close, notwithstanding the extreme sensitivity of the eigenvalues and eigenvectors of  $A$  to small changes in  $a_r, b_r, c_r$ .

When preparing their tables of Lamé polynomials, Arscott and Khabaza checked that the computed elements  $\bar{x}_i$  satisfied the unused equation of the set (12) to within a very small error. It is interesting to observe that this provides a very weak overall check, though it does give some guarantee that the computer has worked correctly. This point is illustrated in Table 3. There we have taken the value  $\bar{\lambda} = 592.0$  and have derived a vector  $\bar{x}$  for the case  $N = 12$ ,  $k^2 = 0.9$  by the process described in Section 3. Now this  $\bar{\lambda}$  is not close to any eigenvalue of  $A$ , though it certainly is an exact eigenvalue of an infinity of neighbouring matrices. The computed vector is therefore in no sense an eigenvector of  $A$ , yet the check appears to work extremely well. The computed vector is of course an accurate eigenvector of any perturbed matrix  $\bar{A}$  which has an eigenvalue close to  $\bar{\lambda}$ . The residual corresponding to the computer vector is as small as that corresponding to the true eigenvalue and eigenvector of Table 2. This is not surprising since 592.0 is certainly the exact eigenvalue of an  $\bar{A}$  that can

Table 3

$N = 12$ ,  $k^2 = 0.9$ ,  $\bar{\lambda} = 592.0$ . Calculation of  $\bar{x}$  using nine significant decimals

FORWARD SEQUENCE	BACKWARD SEQUENCE	NORMALIZED $\bar{x}$
$\bar{x}_0 = (1.00000\ 000)10^0$	$\bar{x}'_{12} = (1.00000\ 000)10^0$	$(4.22800\ 442)10^{-9}$
$\bar{x}_1 = (2.96000\ 000)10^2$	$\bar{x}'_{11} = (5.93853\ 428)10^0$	$(1.25148\ 931)10^{-6}$
$\bar{x}_2 = (1.44602\ 000)10^4$	$\bar{x}'_{10} = (1.54164\ 434)10^1$	$(6.11377\ 896)10^{-5}$
$\bar{x}_3 = (2.75969\ 664)10^5$	$\bar{x}'_9 = (2.29697\ 043)10^1$	$(1.16680\ 096)10^{-3}$
$\bar{x}_4 = (2.71510\ 608)10^6$		$(1.14794\ 805)10^{-2}$
$\bar{x}_5 = (1.57308\ 652)10^7$		$(6.65101\ 676)10^{-2}$
$\bar{x}_6 = (5.76820\ 168)10^7$		$(2.43879\ 822)10^{-1}$
$\bar{x}_7 = (1.39028\ 932)10^8$		$(5.87814\ 939)10^{-1}$
$\bar{x}_8 = (2.23252\ 031)10^8$		$(9.43910\ 575)10^{-1}$
$\bar{x}_9 = (2.36518\ 201)10^8$		$(1.00000\ 000)10^0$
		$(6.71164\ 208)10^{-1}$
		$(2.58537\ 690)10^{-1}$
		$(4.35356\ 062)10^{-2}$
CHECK $a_9\bar{x}_8 + (b_9 - \bar{\lambda})x_9 + c_9\bar{x}_{10} = -278.642402 + 278.642400$ .		

be derived from  $A$  by altering  $b_{10}$  by a quantity of the order of  $10^{-1} \cdot 2^{-16}$ , and the residual obtained from such an  $\bar{A}$  is scarcely different from that obtained from  $A$  itself. If we use any value of  $\bar{\lambda}$  between, say, 592.0 and 593.0 the "check" gives an equally satisfactory result! For larger values of  $N$  the check becomes even weaker.

## References

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## Book Review

*Elements of Numerical Analysis*, by Peter Henrici, 1965; 328 pages. (London and New York: John Wiley and Sons Ltd., 60s.)

Among the many books on numerical analysis which have been published in the last few years this volume is outstanding. Professor Henrici has developed the subject as a mathematical discipline, emphasising the fact that the roots of numerical analysis lie in the field of mathematical analysis, and that numerical analysis is a science and not an art. This book is not a collection of recipes, and potential users requiring this type of approach to the subject will have to look elsewhere. Rather, it is a book for mathematicians and is suitable for a first course in numerical analysis for such students. Among topics which hitherto have not appeared in a book of this type are Romberg integration and, particularly valuable, a nicely written introduction to the theory of error propagation.

It is also a relief to open a book on numerical analysis which does not devote an unnecessarily large amount of

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attention to the theory of finite differences. One topic which has been omitted completely is numerical methods in algebra and matrix theory, but several well written books on this topic are currently available. For most people, a thorough understanding of any branch of mathematics is only obtained after a large number of examples have been worked out. The present volume is excellent in this respect since it has over 300 examples of varying degrees of difficulty, together with a small number of research problems.

The book is divided into three parts: Part one deals with the solution of equations, including simple iteration methods, Bernoulli's method and the Quotient Difference Algorithm. The second part deals with interpolation and approximation and ends with a discussion of numerical solutions of differential equations. The final part is quite short, containing chapters on number systems and error propagation.

The book can be recommended for an introductory course on numerical analysis.

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