algorithm for summing the Chebyshev series

$$\left.\begin{aligned} &b_{n+2} = b_{n+1} = 0 \\ &b_r = 2xb_{r+1} - b_{r+2} + a_r, \qquad r = n, n-1 \ldots 0 \\ &\sum_{r=0}^{n}{}' a_r T_r(x) = \tfrac{1}{2}(b_0 - b_2) \end{aligned}\right\} \quad (5)$$

gives a bound

$$|b_0| \leqslant |a_0| + 2|a_1| + \ldots + (n+1)|a_n|, \quad (6)$$

while that for the power series

$$\left.\begin{aligned} &g_{n+1} = 0 \\ &g_r = xg_{r+1} + f_r, \qquad r = n, n-1 \ldots 0 \\ &\sum_{r=0}^{n} f_r x^r = g_0 \end{aligned}\right\} \quad (7)$$

gives

$$|g_0| \leqslant |f_0| + |f_1| + \ldots + |f_n|. \quad (8)$$

The work of rearrangement is slight, and may be performed either by hand or by a simple computer program; the conversion matrix is readily obtainable from the explicit formulae for the Chebyshev polynomials. On a computer with a fast multiplier the algorithm (7) may run considerably faster than (5)—on KDF 9 the ratio is approximately two to one; and for such a machine the slight labour of rearrangement may be well worth while when evaluating a function whose coefficients are known in advance. When the Chebyshev coefficients are formed during the program itself there is frequently no guarantee that they will be suitable for rearrangement, and hence direct summation by (5) is the safer technique.

Acknowledgements

My thanks are due to Mr. C. W. Clenshaw and Mr. G. F. Miller, of the National Physical Laboratory, for a helpful discussion, and to English Electric-Leo-Marconi Computers Ltd. for permission to publish this paper.

Reference

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Note on the matrix equation $Ax = \lambda Bx$

By Henry E. Fettis*

The equation

$$Ax = \lambda Bx \quad (1)$$

is normally solved by reducing it to an equation of the type

$$Cy = \lambda y \quad (2)$$

for which many standard methods are available. We are concerned here with the case where A and B are both symmetric and B is positive definite.

The most obvious device for reducing equation (1) to equation (2) is to premultiply by B^{-1}, thus arriving at the conventional form with $C = B^{-1}A$. This procedure has the disadvantage that it destroys symmetry, and thus excludes methods which depend on this property of C (e.g., the Jacobi method).

A second method consists of first reducing B to diagonal form by finding an orthogonal matrix T such that

$$TB\tilde{T} = D \quad (3)$$

where D is diagonal. (T may, for example, be determined by the Jacobi method.) Since B is positive definite, $D^{-1/2}$ is real and finite. We may then rewrite (1) as

$$Cy = \lambda y \quad (4)$$

where

$$C = D^{-1/2}TA\tilde{T}D^{-1/2}$$

$$x = \tilde{T}D^{-1/2}y. \quad (5)$$

Since C is still symmetric, equation (4) may be handled by standard methods as before. The amount of work is seen to be somewhat greater than that required for two equations of type (2).

A third method which retains the symmetry of the original equation and involves fewer operations is the following:

(1) By elementary row operations on B, find an upper-triangular matrix S such that SB is lower-triangular. To find S, it is only necessary to perform Gaussian elimination on B, while simultaneously performing the same row operations on the unit matrix.

(2) Since S is lower-triangular, $SB\tilde{S}$ is also; and since $SB\tilde{S}$ is symmetric $SB\tilde{S}$ must be a diagonal matrix, say D. Equation (1) is thus reduced to

$$SA\tilde{S}Z = \lambda DZ$$

from which equation (4) follows as in the second method.

The number of operations required to arrive at the desired form is clearly much less than for either of the first two methods. It is noted that SA may be computed concurrently with S by performing on A the same row operations as were performed on B.

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