A new method of solving second-order differential equations when the first derivative is present

By B. T. Allen*

Differential equations of the form $y'' + f(x, y) = 0$ can be solved using the approximation

$$y_{i+1} - 2y_i + y_{i-1} + \frac{h^2}{12}[f_{i+1} + 10f_i + f_{i-1}] + O(h^6) = 0,$$

obtaining an error of $O(h^4)$ over the complete range. In this note we extend the method to deal with equations of the form $y'' + f(x, y, y') = 0$.

Differential equations of the form $y'' + f(x, y) = 0$ can be approximated so that we obtain the set of relations

$$y_{i+1} - 2y_i + y_{i-1} + \frac{h^2}{12}[f(x_{i+1}, y_{i+1}) + 10f(x_i, y_i) + f(x_{i-1}, y_{i-1})] + O(h^6) = 0, \quad (1)$$

$$\text{for } i = 1, \ldots, n - 1.$$

y_i is the approximation to the solution at the $n + 1$ equidistant tabular points x_i. The solution can then be found by solving these equations, either by recurrence methods, or by using a general n-dimensional equation solver algorithm. This method is normally used for solving boundary value problems, so in our discussion and in our example we will take y_0 and y_n as given. In general the procedure gives rise to an overall error of $O(h^4)$ (for details, see Henrici, 1962). A similar approximation which solves the more general problem $y'' + f(x, y, y') = 0$ to fourth order, does not seem to have been mentioned in the literature. As (1) has been used quite extensively it has seemed worth while to extend the method to the more general problem and to give the details here.

The relation we have derived when the first derivative is present is

$$\begin{aligned} y_{i+1} - 2y_i + y_{i-1} &+ \frac{h^2}{12}[f(x_{i+1}, y_{i+1}, y'^{+}_{i+1}) \\ &+ 4f(x_i, y_i, y'^{0}_i) + f(x_{i-1}, y_{i-1}, y'^{-}_{i-1})] \\ &+ \frac{h^2}{2}f\Big(x_i, y_i, y'^{0}_i + \frac{h}{12}[f(x_{i+1}, y_{i+1}, y'^{0}_{i+1}) \\ &- f(x_{i-1}, y_{i-1}, y'^{-}_{i-1})]\Big) + O(h^6) = 0, \quad (2) \end{aligned}$$

where

$$y'^{+}_{i+1} = (y_{i-1} - 4y_i + 3y_{i+1})/2h,$$
$$y'^{0}_i = (y_{i+1} - y_{i-1})/2h$$

and

$$y'^{-}_{i-1} = (-3y_{i-1} + 4y_i - y_{i+1})/2h.$$

This requires four evaluations of the function each time the equation is to be evaluated, as compared with three for (1). It can readily be seen that (2) reduces to (1) when y' is absent. The simplest way to show that the error is of order h^6 seems to be to expand the various terms. In doing this we assume that the various derivatives and partial derivatives of y and f we will require, exist and are continuous. We shall use the notation that a subscript on f means partial differention with respect to that variable whose position is specified by the subscript. Thus f_3 denotes $\frac{\partial f}{\partial y'}$. The operators μ and δ have their usual "mean sum" and "central difference" meaning. Then

$$\begin{aligned} f(x_{i+1}, y_{i+1}, y'^{+}_{i+1}) &= f\Big(x_{i+1}, y_{i+1}, y'_{i+1} - \frac{1}{h}\Big(\frac{1}{3}\mu\delta^3 + \frac{1}{12}\delta^4 - \frac{1}{20}\mu\delta^5\Big)y_i + O(h^5)\Big), \\ &= f(x_{i+1}, y_{i+1}, y'_{i+1}) - \frac{1}{h}\Big[\Big(\frac{1}{3}\mu\delta^3 + \frac{1}{12}\delta^4 - \frac{1}{20}\mu\delta^5\Big)y_i\Big]f_3(x_{i+1}, y_{i+1}, y'_{i+1}) + O(h^5). \end{aligned}$$

$$f(x_i, y_i, y'^{0}_i) = f(x_i, y_i, y'_i) + \frac{1}{h}\Big[\Big(\frac{1}{6}\mu\delta^3 - \frac{1}{30}\mu\delta^5\Big)y_i\Big]f_3(x_i, y_i, y'_i) + O(h^6).$$

$$f(x_{i-1}, y_{i-1}, y'^{-}_{i-1}) = f(x_{i-1}, y_{i-1}, y'_{i-1}) - \frac{1}{h}\Big[\Big(\frac{1}{3}\mu\delta^3 - \frac{1}{12}\delta^4 - \frac{1}{20}\mu\delta^5\Big)y_i\Big]f_3(x_{i-1}, y_{i-1}, y'_{i-1}) + O(h^5).$$

* *School of Physics, University of Sydney, Sydney, N.S.W., Australia.*

Thus

$$f(x_{i+1}, y_{i+1}, y_{i+1}^{\prime +}) + 4f(x_i, y_i, y_i^{\prime 0}) + f(x_{i-1}, y_{i-1}, y_{i-1}^{\prime -})$$
$$= [f(x_{i+1}, y_{i+1}, y_{i+1}') + 4f(x_i, y_i, y_i') + f(x_{i-1}, y_{i-1}, y_{i-1}')]$$
$$- \frac{1}{h}\left[\frac{1}{3}\mu\delta^3 y_i \,.\, \delta^2 f_3 + \frac{1}{6}\delta^4 y_i \,.\, \mu\delta f_3 + \frac{1}{30}\mu\delta^5 y_i \,.\, f_3\right] + O(h^6).$$

Also

$$y_i' = \frac{1}{h}\left(\mu\delta - \frac{1}{6}\mu\delta^3 + \frac{1}{30}\mu\delta^5\right)y_i + O(h^6),$$
$$= y_i^{\prime 0} - \frac{1}{6h}\mu\delta\left(h^2 y_i'' + \frac{1}{12}\delta^4 y_i + O(h^6)\right) + \frac{1}{30h}\mu\delta^5 y_i + O(h^6),$$
$$= y_i^{\prime 0} + \frac{h}{12}[f(x_{i+1}, y_{i+1}, y_{i+1}') - f(x_{i-1}, y_{i-1}, y_{i-1}')] + \frac{7}{360h}\mu\delta^5 y_i + O(h^6),$$
$$= y_i^{\prime 0} + \frac{h}{12}[f(x_{i+1}, y_{i+1}, y_{i+1}^{\prime +}) - f(x_{i-1}, y_{i-1}, y_{i-1}^{\prime -})] + \frac{1}{18}\mu\delta^3 y_i \,.\, \mu\delta f_3 + \frac{1}{72}\delta^4 y_i \,.\, \mu f_3 + \frac{7}{360h}\mu\delta^5 y_i + O(h^6).$$

The error is thus of order h^6. We could have found and given the error in terms of derivatives instead of differences with very little change in working.

Exactly the same considerations now apply for solving the problem with (2) as applied with (1). The set of relations can be solved as a set of non-linear equations or as a recurrence relation. To solve as a general non-linear set of equations, although less likely to give rise to stability difficulties, is expensive in time and we did not consider it. We used the recurrence method as we now describe it. The method is started by guessing a value of y_1 and then continued by solving the relations (2) till y_n is found. y_n will presumably be incorrect, but then further guesses, using a regula-falsi algorithm are made to y_1 until the correct value of y_n is obtained. As the relation (2) is implicit a non-linear equation must be solved at each stage to find y_{i+1}. The best way to solve this seemed to be to make a first estimate by extrapolation on previous values of y_i, make a second estimate by iteration, and whatever further estimates necessary by regula-falsi. If errors grow in the recurrence relation due to mathematical instability it is sometimes possible to overcome this by recurring in the opposite direction.

Apart from fairly trivial examples to check programming, this method was tried and tested on the equation

$$y''(3 + \sin \pi(0\cdot 5 - x)) - \pi \cos \pi(0\cdot 5 - x)(y' + 3) + \frac{4}{3}\left(\frac{4}{3}y\right)^3 = 0,\ y(0) = 1,\ y(1) = 0.$$

We had solved this problem already by some calculus-of-variations methods and thus had an independent solution to check against. The results are shown in **Table 1** and are what we might expect.

Therefore

$$y_{i+1} - 2y_i + y_{i-1} + \frac{h^2}{12}[f(x_{i+1}, y_{i+1}, y_{i+1}^{\prime +}) + 4f(x_i, y_i, y_i^{\prime 0}) + f(x_{i-1}, y_{i-1}, y_{i-1}^{\prime -})]$$
$$+ \frac{h^2}{2}f\left(x_i, y_i, y_i^{\prime 0} + \frac{h}{12}[f(x_{i+1}, y_{i+1}, y_{i+1}^{\prime +}) - f(x_{i-1}, y_{i-1}, y_{i-1}^{\prime -})]\right)$$
$$= y_{i+1} - 2y_i + y_{i-1} + \frac{h^2}{12}[f(x_{i+1}, y_{i+1}, y_{i+1}') + 10f(x_i, y_i, y_i') + f(x_{i-1}, y_{i-1}, y_{i-1}')]$$
$$- \frac{h}{12}\left[\frac{1}{3}\mu\delta^3 y_i \,.\, \delta^2 f_3 + \frac{1}{6}\delta^4 y_i \,.\, \mu\delta f_3 + \frac{1}{30}\mu\delta^5 y_i \,.\, f_3\right]$$
$$- \frac{h^2}{2}\left[\frac{1}{18}\mu\delta^3 y_i \,.\, \mu\delta f_3 + \frac{1}{72}\delta^4 y_i \,.\, \mu f_3 + \frac{7}{360h}\mu\delta^5 y_i\right] f_3 + O(h^8),$$
$$= -\frac{1}{240}\delta^6 y_i - h\left[\frac{1}{36}\mu\delta^3 y_i \,.\, \delta^2 f_3 + \frac{1}{72}\delta^4 y_i \,.\, \mu\delta f_3 + \frac{1}{80}\mu\delta^5 y_i \,.\, f_3\right]$$
$$- h^2 f_3\left[\frac{1}{36}\mu\delta^3 y_i \,.\, \mu\delta f_3 + \frac{1}{144}\delta^4 y_i \,.\, f_3\right] + O(h^8).$$

Table 1

The value and error of the solution at $x = 0 \cdot 2$ and $0 \cdot 5$

VALUE OF h	SOLUTION AT $0 \cdot 2$	ERROR AT $0 \cdot 2$	SOLUTION AT $0 \cdot 5$	ERROR AT $0 \cdot 5$
0·1	0·67621 94	−0·00000 47	0·27331 2	−0·00001 4
0·05	0·67622 38	−0·00000 03	0·27332 56	−0·00000 09
0·025	0·67622 41	0	0·27332 65	−0·00000 005

There seems to be an obvious gain in using this method rather than a second order method, such as

$$y_{i+1} - 2y_i + y_{i-1} + h^2 f(x_i, y_i, (y_{i+1} - y_{i-1})/2h) = O(h^4), \text{ for } i = 1, \ldots, n-1.$$

In our method we have obtained an error of $O(h^4)$ instead of $O(h^2)$ with only four times as much work. h^2 extrapolation could indeed be used on second order methods but we could use h^4 extrapolation on our own methods, and it would seem that the nearer we started to the true solution before we introduced the instability possibilities of extrapolation the better. In comparison with explicit methods of solving the problem the method seems less efficient. A Runge–Kutta method for instance which obtains the same order of accuracy needs only four function evaluations per step. We usually need to evaluate the recurrence relation twice to find y_{i+1} with sufficient accuarcy, and thus need eight function evaluations per step. There may, however, be circumstances when explicit methods lead to unnecessary instabilities and where the present method might be of value.

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Book Review

Sampling Systems Theory and its Application, Volumes 1 *and* 2, by Ya. Z. Tsypkin, 1964; 742 pages. (Oxford: *Pergamon Press Ltd.*, 100s. per volume).

The author of these two volumes works at the *Academy of Sciences* of the USSR and is an acknowledged expert in this field, having published several papers in recent years. The two volumes are entirely complementary and require no prior knowledge of sampling systems.

The first chapter classifies various pulse systems, and gives practical examples. It is possible that benefit could have been gained if a little of the fine detail could have been sacrificed for a discussion in a broader context of fields where pulse systems are inevitable and, in fields where this is not so, why they may be preferred to continuous systems. Chapter II lays the basic theoretical groundwork and provides a valuable reference provided one does not object to unfamiliar terminology, *e.g.* the use throughout of the discrete Laplace transformation rather than the Z-transform. This chapter is very thorough, although it would have been of advantage if the important results could have been summarized more effectively. Application of the techniques to solving difference equations is included. Chapter III is concerned with the application of the theory to open-loop systems.

Chapter V, in the second volume, considers closed-loop systems and again covers a lot of ground. It is here that some of the author's original work is contained and, therefore, this section is likely to be of most interest to those already proficient in the subject.

Much of the value of these volumes stems from the many wide and varied problems on both closed- and open-loop pulse systems. The bibliography and tables contained in the appendix are also very useful. Of particular interest to those associated with digital computers is their use in control systems; this receives adequate attention.

To summarize, these two volumes certainly merit attention from scientists specializing in sampled-data systems.

G. F. TURNBULL