A lower estimate of the cumulative truncation error in Milne's method

By A. C. Smith*

A method is given for the estimation of the cumulative truncation error which may be substantially less than the usual bound.

1. Introduction

In the numerical integration of the differential equation

$$y' = f(x, y)\left\{\begin{array}{l} x = x_0 \\ y = y_0 \end{array}\right. \quad (1.1)$$

using Milne's predictor-corrector method:

$$y_{n+1}(p) = y_{n-3} + \frac{4h}{3}(2y'_{n-2} - y'_{n-1} + 2y'_n) + \frac{28}{90}h^5 y^{(5)}(s_1) \quad (1.2)$$

$$y_{n+1}(c) = y_{n-1} + \frac{h}{3}(y'_{n-1} + 4y'_n + y'_{n+1}) - \frac{h^5}{90}y^{(5)}(s_2) \quad (1.3)$$

a bound for the magnitude of the cumulative error due to truncation is given (see Milne, 1953), by

$$E_n = \frac{h^4 M}{180G}\left[\left(\frac{1 + hG}{1 - hG/3}\right)^n - 1\right] \quad (1.4)$$

where n is the number of steps, G the maximum value of $\left|\frac{\partial f}{\partial y}\right|$ and M the maximum value of $|y^{(5)}|$. G is estimated from the values of $\left(\frac{y'_{n+1} - y'_n}{y_{n+1} - y_n}\right)$ obtained during the computation, and M from the values of

$$y(p) - y(c) \doteqdot \frac{29}{90}h^5 y^{(5)}(s_3). \quad (1.5)$$

E_n is obtained by solving a difference equation with constant coefficients which dominates the difference equation for the cumulative error (see (2.3) below). The bound E_n is frequently very conservative.

2. Asymptotic solution of the difference equation

Using Milne's notation let y_n be the calculated, and z_n the true value of the solution.
Then

$$z_{n+1} = z_{n-1} + \frac{h}{3}(z'_{n-1} + 4z'_n + z'_{n+1}) + T_n \quad (2.1)$$

$$y_{n+1} = y_{n-1} + \frac{h}{3}(y'_{n-1} + 4y'_n + y'_{n+1}) \quad (2.2)$$

where $T_n = -\frac{h^5}{90}y^{(5)}$.

The cumulative truncation error after n steps, $e_n = y_n - z_n$, satisfies the difference equation

$$e_{n+1} = e_{n-1} + \frac{h}{3}(e_{n-1}g_{n-1} + 4e_n g_n + e_{n+1}g_{n+1}) - T_n \quad (2.3)$$

where

$$g = \frac{\partial f}{\partial y} \text{ and } g_n = g(x_n, \bar{y}_n),$$

$\bar{y}_n$ lying between y_n and z_n.

Let
$$S_{2r}(g) = \frac{1}{3}(g_0 + 4g_1 + 2g_2 + \ldots + 4g_{2r-1} + g_{2r}) \quad (2.4)$$

$$S_{2r+1}(g) = \frac{1}{3}(g_1 + 4g_2 + 2g_3 + \ldots + 4g_{2r} + g_{2r+1}) \quad (2.5)$$

be the even and odd "Simpson" sums of g_n.
Then

$$hS_{2n}(g) = \int_0^{2nh} g(x)dx + \frac{nh^5}{90}g^{(4)}(t_1) \quad (2.6)$$

$$hS_{2n+1}(g) = \int_h^{(2n+1)h} g(x)dx + \frac{nh^5}{90}g^{(4)}(t_2). \quad (2.7)$$

Moreover

$$\frac{1}{2}h(g_0 + g_1) = \int_0^h g(x)dx + \frac{h^3}{12}g^{(2)}(t_3) \quad (2.8)$$

$$\frac{1}{2}h(g_{2n} + g_{2n+1}) = \int_{2nh}^{(2n+1)h} g(x)dx + \frac{h^3}{12}g^{(2)}(t_4). \quad (2.9)$$

Hence it follows that

$$hS_{2n}(g) + \frac{1}{2}h(g_{2n} + g_{2n+1}) = \frac{1}{2}h(g_0 + g_1) + hS_{2n+1}(g) + O(h^3). \quad (2.10)$$

Consider now the difference equation

$$Y_{n+1} - Y_{n-1} = \frac{h}{3}(g_{n+1}Y_{n+1} + 4g_n Y_n + g_{n-1}Y_{n-1}) \quad (2.11)$$

* *Department of Mathematics, University of Windsor, Ontario, Canada.*

When $h=0$, the complementary function is $A_0+B_0(-1)^n$, where A_0 and B_0 are arbitrary constants.

Next let

$$Y_n = A_0 + hA_1(n) + h^2A_2(n) + h^3A_3(n) + \dots \quad (2.12)$$

Substituting in (2.11) and equating coefficients of like powers of h we obtain

$$\left.\begin{aligned} A_1(2n) &= A_1(0) + A_0S_{2n}(g) \\ A_1(2n+1) &= A_1(1) + A_0S_{2n+1}(g) \end{aligned}\right\} \quad (2.13)$$

$$\left.\begin{aligned} A_2(2n) &= A_2(0) + S_{2n}(gA_1) \\ A_2(2n+1) &= A_2(1) + S_{2n+1}(gA_1) \\ &\text{etc.} \end{aligned}\right\} \quad (2.14)$$

Let $A_1(0) = 0$ and $A_1(1) = \frac{1}{2}(g_0 + g_1)A_0$.

Then

$$hA_1(2n) = A_0hS_{2n}(g)$$

$$\begin{aligned} hA_1(2n+1) &= A_0\left[\frac{h}{2}(g_0+g_1) + hS_{2n+1}(g)\right] \\ &= A_0\left[hS_{2n}(g) + \frac{h}{2}(g_{2n}+g_{2n+1})\right] \\ &\quad + O(h^3) \text{ in virtue of (2.10).} \end{aligned} \quad (2.15)$$

Hence, whether n is even or odd we may write

$$hA_1(n) = A_0\int_0^{nh} g(x)dx + O(h^3). \quad (2.16)$$

Again

$$\begin{aligned} h^2S_{2n}(gA_1) &= \frac{h^2}{3}[g_0A_1(0) + 4g_1A_1(1) + \dots \\ &\quad \dots + 4g_{2n-1}A_1(2n-1) + g_{2n}A_1(2n)] \\ &= \frac{1}{3}A_0h\left[4g_1\int_0^h g(x)dx + 2g_2\int_0^{2h} g(x)dx + \dots\right. \\ &\quad \left.\dots + g_{2n}\int_0^{2nh} g(x)dx + O(nh^4)\right] \\ &= A_0\int_0^{2nh} g(u)du\int_0^u g(x)dx + O(h^3) \\ &= \frac{1}{2}A_0\left[\int_0^{2nh} g(x)dx\right]^2 + O(h^3) \\ &= \frac{1}{2}A_0h^2S_{2n}^2(g) + O(h^3) \end{aligned} \quad (2.17)$$

Take $A_2(0) = 0,\ h^2A_2(1) = \frac{1}{2}A_0\left[\frac{1}{2}h(g_0+g_1)\right]^2$

Then $$h^2A_2(2n) = \frac{1}{2}A_0h^2S_{2n}^2(g) + O(h^3)$$

$$h^2A_2(2n+1) = \frac{1}{2}A_0h^2\left[\frac{1}{2}(g_0+g_1) + S_{2n+1}(g)\right]^2 + O(h^3). \quad (2.18)$$

The foregoing results suggest that

$$Y_n = A_0 \exp[hR_n(g)] \quad (2.19)$$

may be a solution of the difference equation (2.11), where

$$\left.\begin{aligned} R_{2n}(g) &= \frac{1}{3}(g_0 + 4g_1 + 2g_2 + \dots \\ &\quad \dots + 4g_{2n-1} + g_{2n}) \\ R_{2n+1}(g) &= \frac{1}{3}(g_1 + 4g_2 + 2g_3 + \dots \\ &\quad \dots + 4g_{2n} + g_{2n+1}) + \frac{1}{2}(g_0+g_1). \end{aligned}\right\} \quad (2.20)$$

Substituting (2.19) in (2.11) and using (2.10) it can be seen that the residue is $O(h^2)$.

Similarly the solution corresponding to $B_0(-1)^n$ is found to be

$$\left.\begin{aligned} &(-1)^nB_0 \exp[hR_n(k)] \\ \text{where} \quad &k_n = (-1)^ng_n \end{aligned}\right\} \quad (2.21)$$

Thus the complementary function of (2.11) is

$$A_0 \exp[hR_n(g)] + (-1)^nB_0 \exp[hR_n(k)] + O(h^2). \quad (2.22)$$

If we assume that T_n in (2.3) can be expressed in the form

$$C_0\theta^n + D_0 \quad \text{where } \theta^2 \neq 1$$

and substitute

$$[C_0 + hC_1(n) + h^2C_2(n) + \dots]\frac{\theta^{n+1}}{\theta^2-1} \quad (2.23)$$

and $$[D_0 + hD_1(n) + h^2D_2(n) + \dots]\frac{1}{2}n$$

it is found that the particular solution

$$\begin{aligned} &-C_0\left[\frac{\theta^{n+1}-\theta}{\theta^2-1} + \frac{\theta}{\theta^2-1}\exp[hR_n(p)]\right] \quad (2.24) \\ &-D_0\left[\frac{1}{2}n - 1 + \exp[hR_n(q)]\right] \end{aligned}$$

where $$p_n = \theta^ng_n \text{ and } q_n = \frac{1}{2}ng_n$$

satisfies the difference equation (2.3) with residue $O(h^2)$.

The complete solution of (2.3) can then be written as

$$\begin{aligned} e_n &= A_0 \exp[hR_n(g)] + (-1)^nB_0 \exp[hR_n(k)] \\ &\quad - C_0\left[\frac{\theta^{n+1}-\theta}{\theta^2-1} + \frac{\theta}{\theta^2-1}\exp[hR_n(p)]\right] \\ &\quad - D_0\left[\frac{1}{2}n - 1 + \exp[hR_n(q)]\right] + O(h^2). \end{aligned} \quad (2.25)$$

A_0 and B_0 are arbitrary constants which can be determined on the assumption that

$$e_0 = e_1 = 0.$$

3. Example (See Nielsen, 1956)

$$y' = \frac{2x-1}{x^2}y + 1 \qquad \begin{cases} x_0 = 1 \\ y_0 = 2 \end{cases} \qquad (3.1)$$

$h = 0 \cdot 1$

n	x	y	$T \times 10^{10}$	g
0	1·0	2·0000 0000		1·0600
1	1·1	2·3148 5619	−80482	0·9392
2	1·2	2·6589 3596	−36009	0·8409
3	1·3	3·0317 3475	−15908	0·7650
4	1·4	3·4328 9892	−8946	0·7039
5	1·5	3·8622 0178	−4804	0·6532
6	1·6	4·3194 6504	−3175	0·6101
7	1·7	4·8045 7482	−1675	0·5729
8	1·8	5·3174 3008	−1346	0·5403
9	1·9	5·8579 6919	−627	0·5115
10	2·0	6·4261 2946	−668	0·4858
11	2·1	7·0218 7604	−226	0·4627
12	2·2	7·6451 6696	−391	0·4417

The exact solution is $y = x^2\left(1 + \exp\left(\frac{1}{x} - 1\right)\right)$. (3.2)

Substitution in (1.4) gives $E_{12} = 14{,}100 \times 10^{-8}$ (3.3)

Approximating T_n by $-146 \times 10^{-7} \times \left(\frac{1}{2}\right)^n$ and assuming that $e_0 = e_1 = 0$, the values of e_2, e_3 and e_4 calculated from (2.3) are found to be

$$\left.\begin{aligned} e_2 &= 75{,}105 \times 10^{-10} \\ e_3 &= 49{,}096 \times 10^{-10} \\ e_4 &= 102{,}568 \times 10^{-10}. \end{aligned}\right\} \qquad (3.4)$$

Putting $n = 2$ and 4 in (2.25) and equating to e_2 and e_4 in (3.4) we find that

$$\left.\begin{aligned} A_0 &= 2{,}987 \times 10^{-8} \\ B_0 &= -\ 3{,}412 \times 10^{-8}. \end{aligned}\right\} \qquad (3.5)$$

Substituting for A_0 and B_0 in (2.25) with $n = 12$ we obtain

$$e_{12} = 3{,}762 \times 10^{-8}. \qquad (3.6)$$

The value of (3.2) when $x = 2 \cdot 2$ is $7 \cdot 6451\ 5888$. The actual error is thus 808×10^{-8}.

References

Milne, W. E. (1953). *Numerical Solution of Differential Equations*, Wiley: New York, pp. 66–69.

Nielsen, K. L. (1956). *Methods in Numerical Analysis*, Macmillan: New York, pp. 230–232.

Book Review

Optimization Theory and the Design of Feedback Control Systems, by C. W. Merriam III, 1964; 390 pages. (Maidenhead: *McGraw-Hill Publishing Company Ltd.*, 112s.).

This book, by a well-known figure in both academic and industrial research into advanced control problems, is a thorough introduction to Optimal Control Theory as applied to continuous dynamic systems. Its clear exposition starts with the simplest problem, that of choosing the best parameters for a fixed controller, and continues through optimum linear, and linear optimum systems to the design of non-linear control systems.

Full explanations of each point are given, aided by simple worked examples, with illustrated solutions. Included is an account of the application of variational calculus, dynamic programming, and the "maximum principle" to control problems, showing the relation and the differences between the three methods. There is an enlightening chapter on the numerical solution of the two-point boundary problem.

The appendices, apart from a summary of the extensive basic notations and a survey of relevant literature, include short articles on the essential aspects of random signal theory, and on the implications of the state vector description of dynamic systems. There is a fully worked-out example of a linear optimum control system for the aircraft landing problem, and notes on computer methods of numerical integration of differential equations and on the approximation of the Hamiltonian function. Not least is a set of problems based on the content of each chapter.

Based on a post-graduate lecture course, and especially suitable as an advance text, the book will also prove valuable to the practising engineer and mathematician who must turn optimal theory into working control schemes. Such applications will provide employment for man and computer for a good many years—indeed as long as there are processes to control, and computers with which to do so.

P. A. N. Briggs