A Runge-Kutta method for the numerical solution of the Goursat problem in hyperbolic partial differential equations

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A Runge-Kutta type method is developed for the numerical solution of second order hyperbolic partial differential equations. Numerical examples of the method are given.

In this paper we consider a numerical method for the solution of the Goursat problem

$$\begin{aligned} &u_{xy} = f(x, y, u, u_x, u_y) \\ &u(x, 0) = \sigma(x),\ u(0, y) = \tau(y),\ \sigma(0) = \tau(0) \\ &0 \leqslant x \leqslant a,\ 0 \leqslant y \leqslant b. \end{aligned} \tag{1}$$

The numerical solution of (1) over a set $D\{(x, y): 0 \leqslant x \leqslant a,\ 0 \leqslant y \leqslant b\}$ is to be carried out in a stepwise manner over a square mesh on D. The object of our method is to calculate u, u_x, u_y at $(x_0 + h, y_0 + h)$, given u, u_x, u_y at (x_0, y_0), $(x_0 + h, y_0)$, $(x_0, y_0 + h)$.

In this discussion it is assumed that a solution to the above problem exists and is unique (Kamke, 1947 or Jeffrey and Taniuti, 1964) and that f, σ, and τ are sufficiently regular for the subsequent derivations to hold true. It is also assumed that $u_x(0, y)$ and $u_y(x, 0)$ have been calculated along the initial data lines. Techniques on how this can be done are discussed by Moore (1961).

For convenience of notation the following symbolism is used. We denote the value of the function f evaluated at the point (x_0, y_0) by f_1. In a similar manner we use f_2 and f_3 for the values at the respective points $(x_0 + h, y_0)$, $(x_0, y_0 + h)$.

To derive the method under consideration the differential equation is converted into a system of integral equations. After integration we have

$$u(x_0 + h, y_0 + h) = u(x_0 + h, y_0) + u(x_0, y_0 + h) - u(x_0, y_0) + \int_{x_0}^{x_0+h} \int_{y_0}^{y_0+h} f(x, y, u, u_x, u_y)\,dx\,dy \tag{2}$$

$$u_x(x_0 + h, y_0 + h) = u_x(x_0 + h, y_0) + \int_{y_0}^{y_0+h} f(x_0 + h, u(x_0 + h, y), u_x(x_0 + h, y), u_y(x_0 + h, y))\,dy \tag{3}$$

$$u_y(x_0 + h, y_0 + h) = u_y(x_0, y_0 + h) + \int_{x_0}^{x_0+h} f(x, y_0 + h, u(x, y_0 + h), u_x(x, y_0 + h), u_y(x, y_0 + h))\,dx. \tag{4}$$

If we approximate the double integral in (2) by means of the trapezoidal rule for double integrals (Runge–Willers, 1915)

$$\begin{aligned} \int_{x_0}^{x_0+h} \int_{x_0}^{y_0+h} F(x, y)\,dx\,dy = {} & h^2[F(x_0, y_0) + F(x_0 + h, y_0) \\ & + F(x_0, y_0 + h) + F(x_0 + h, y_0 + h)]/4 \\ & - h^4[F_{xx}(t, y_0) + F_{xx}(t, y_0 + h) + F_{yy}(x_0, \eta) \\ & + F_{yy}(x_0 + h, \eta)]/24 + h^6[F_{xxyy}(t, \lambda)]/144 \end{aligned} \tag{5}$$

(here $x_0 < t < x_0 + h$, $y_0 < \eta < y_0 + h$, $y_0 < \lambda < y_0 + h$)

we obtain

$$\begin{aligned} u(x_0 + h, y_0 + h) = {} & u(x_0 + h, y_0) + u(x_0, y_0 + h) \\ & - u(x_0, y_0) + h^2[f_1 + f_2 + f_3 + f(x_0 + h, y_0 + h, \\ & u(x_0 + h, y_0 + h), u_x(x_0 + h, y_0 + h), \\ & u_y(x_0 + h, y_0 + h))]/4 - h^4[f_{yy}(x_0 + h, \eta, \\ & u(x_0 + h, \eta), u_x(x_0 + h, \eta), u_y(x_0 + h, \eta)) \\ & + f_{yy}(x_0, \eta, u(x_0, \eta), u_x(x_0, \eta), u_y(x_0, \eta)) \\ & + f_{xx}(t, y_0, u(t, y_0), u_x(t, y_0), u_y(t, y_0)) + f_{xx}(t, y_0 + h, \\ & u(t, y_0 + h), u_x(t, y_0 + h), u_y(t, y_0 + h))]/24 \\ & + h^6[f_{xxyy}(t, \lambda, u(t, \lambda), u_x(t, \lambda), u_y(t, \lambda))]/144. \end{aligned} \tag{6}$$

Here

$$x_0 < t < x_0 + h,\ y_0 < \eta < y_0 + h,\ y_0 < \lambda < y_0 + h.$$

The reader should note that in the right-hand side of (6) u, u_x, u_y are not known. Approximate values for these quantities are obtained in the succeeding discussion.

By Taylor expansions the following estimate for u at $(x_0 + h, y_0 + h)$ can be obtained:

$$\begin{aligned} u(x_0 + h, y_0 + h) = {} & u(x_0 + h, y_0) + u(x_0, y_0 + h) \\ & - u(x_0, y_0) + h^2 f_1 + h^3[u_{xxy}(x_0 + \theta_1 h, y_0 + \theta_1 h) \\ & + u_{xyy}(x_0 + \theta_1 h, y_0 + \theta_1 h]/6 + O(h^4). \end{aligned}$$

Here $0 < \theta_1 < 1$.

As a useful notation, let u_p be defined as the quantity

$$u_p = u(x_0 + h, y_0) + u(x_0, y_0 + h) - u(x_0, y_0) + h^2 f_1.$$

In order to obtain estimates for u_x and u_y at $(x_0 + h, y_0 + h)$ we first approximate (3) and (4) by the trapezoidal rule to obtain equations (7) and (8).

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$$u_x(x_0 + h, y_0 + h) = u_x(x_0 + h, y_0)$$
$$+ h[f_2 + f(x_0 + h, y_0 + h, u(x_0 + h, y_0 + h), u_x(x_0 + h, y_0 + h), u_y(x_0 + h, y_0 + h))]/2$$
$$- \frac{h^3}{12}\left[\frac{\partial^2}{\partial y^2} f(x_0 + h, y_0 + \theta_3 h, u(x_0 + h, y_0 + \theta_3 h), u_x(x_0 + h, y_0 + \theta_3 h), u_y(x_0 + h, y_0 + \theta_3 h))\right]. \quad (7)$$

(Here $0 < \theta_3 < 1$.)

$$u_y(x_0 + h, y_0 + h) = u_y(x_0, y_0 + h)$$
$$+ h[f_3 + f(x_0 + h, y_0 + h, u(x_0 + h, y_0 + h), u_x(x_0 + h, y_0 + h), u_y(x_0 + h, y_0 + h))]/2$$
$$- \frac{h^3}{12}\left[\frac{\partial^2}{\partial x^2} f(x_0 + \theta_4 h, y_0 + h, u(x_0 + \theta_4 h, y_0), u_x(x_0 + \theta_4 h, y_0), u_y(x_0 + \theta_4 h, y_0))\right]. \quad (8)$$

(Here $0 < \theta_4 < 1$.)

Estimates for u_x and u_y at $(x_0 + h, y_0 + h)$ occurring in the right-hand side of (7) and (8) are obtained by the Taylor expansions

$$u_x(x_0 + h, y_0 + h) = u_x(x_0 + h, y_0) + hf_2 + \frac{h^2}{2} u_{xyy}(x_0 + h, y_0 + \theta_5 h), \quad (9)$$

$$u_y(x_0 + h, y_0 + h) = u_y(x_0, y_0 + h) + hf_3 + \frac{h^2}{2} u_{yxx}(x_0 + \theta_6 h, y_0 + h), \quad (10)$$

where $0 < \theta_5 < 1, 0 < \theta_6 < 1$.

Let $u_{xp} = u_x(x_0 + h, y_0) + hf_2$,

$u_{yp} = u_y(x_0, y_0 + h) + hf_3$.

Approximate values $\tilde{u}_x, \tilde{u}_y$ for u_x and u_y at $(x_0 + h, y_0 + h)$ can be obtained by substituting u_p, u_{xp}, u_{yp} into (7) and (8) and disregarding the truncation error term of (7) and (8), i.e.

$$\tilde{u}_x = u_x(x_0 + h, y_0) + h[f_2 + f(x_0 + h, y_0 + h, u_p, u_{xp}, u_{yp})]/2 \quad (11)$$

$$\tilde{u}_y = u_y(x_0, y_0 + h) + h[f_3 + f(x_0 + h, y_0 + h, u_p, u_{xp}, u_{yp})]/2. \quad (12)$$

It can be shown that

$$u_x(x_0 + h, y_0 + h) - \tilde{u}_x = - \frac{h^3}{12}\left[\frac{\partial^2}{\partial y^2}[f(x_0 + h, y_0 + \theta_3 h, u(x_0 + h, y_0 + \theta_3 h), u_x(x_0 + h, y_0 + \theta_3 h), u_y(x_0 + h, y_0 + \theta_3 h))\right] + \frac{h^3}{24}\left[\frac{\partial f}{\partial u_x} u_{xyy}(x_0 + h, y_0 + \theta_5 h) + \frac{\partial f}{\partial u_y} u_{yxx}(x_0 + \theta_6 h, y_0 + h)\right] + O(h^4), \quad (13)$$

where the partial derivatives of f with respect to u_x and u_y are to be evaluated at $(x_0 + h, y_0 + h)$.

A similar estimate for $u_y(x_0 + h, y_0 + h) - \tilde{u}_y$ can be obtained.

Replacing u, u_x, u_y at $(x_0 + h, y_0 + h)$ in (6) by the respective quantities $u_p, \tilde{u}_x, \tilde{u}_y$ and disregarding the error term in (6) gives us an approximate value u_4 for $u(x_0 + h, y_0 + h)$.

$$u_4 = u(x_0 + h, y_0) + u(y_0, x_0 + h) - u(x_0, y_0) + h^2[f_1 + f_2 + f_3 + f(x_0 + h, y_0 + h, u_p, \tilde{u}_x, \tilde{u}_y)]/4. \quad (14)$$

It can be shown that

$$u(x_0 + h, y_0 + h) - u_4 = - h^4[f_{yy} + f_{xx}]_{(x_0, y_0)}/12 + O(h^5). \quad (15)$$

In practice the writer has found that recalculation of the estimates for u_x and u_y at $(x_0 + h, y_0 + h)$ by the formulae

$$\hat{u}_x = u_x(x_0 + h, y_0) + h[f_2 + f(x_0 + h, y_0 + h, u_4, \tilde{u}_x, \tilde{u}_y)]/2 \quad (16)$$

$$\hat{u}_y = u_y(x_0, y_0 + h) + h[f_3 + f(x_0 + h, y_0 + h, u_4, \tilde{u}_x, \tilde{u}_y)]/2 \quad (17)$$

appears to give better results than using (11) and (12).

The above discussion may appear somewhat complicated; however, on a large-scale digital computer its implementation offers few difficulties. The procedure is summarized here:

Calculate f_1, f_2, f_3—will be stored from previous work.
Calculate u_p
Calculate u_{xp} and u_{yp}
Calculate $\tilde{u}_x$ and $\tilde{u}_y$
Calculate u_4 from (14)
Calculate $\hat{u}_x$, $\hat{u}_y$ from (16) and (17)
Proceed to the next step.

A FORTRAN program for the method under consideration was written (CDC 1604 computer), and the following results were obtained for three computational examples.

In the examples considered below, "error," is to be understood to mean the relative error, i.e.

error = |(true value = approximate value)/true value|.

The first example is the differential equation $u_{xy} = e^{2u}$; with initial conditions

$$u(x, 0) = x/2 - \log(1 + e^x)$$
$$u(0, y) = y/2 - \log(1 + e^y).$$

The solution of this problem is

$$u(x, y) = (x + y)/2 - \log(e^x + e^y).$$

Taking h as 0·05 errors for u were obtained as shown in **Table 1.**

The second example is the differential equation $u_{xy} = u_x \,.\, u_y/u$ with initial data taken along the lines $x = 1$ and $y = 1$; i.e.

$$u(x, 1) = e^{(x+1)} \sin(1)$$
$$u(1, y) = e^{(1+y)} \sin(y).$$

The solution of this problem is

$$u(x, y) = e^{(x+y)} \sin(y).$$

Taking h as 0·05 the errors shown in **Table 2** were obtained.

Example three is $u_{xy} = (u_x + u_y + u)/3$ with initial data taken along the line $x = 0$, $y = 0$, i.e.

$$u(x, 0) = e^x, u(0, y) = e^y.$$

The solution of this problem is

$$u(x, y) = e^{(x+y)}.$$

Taking h as 0·05 the errors shown in **Table 3** were obtained.

Thus the method appears to give satisfactory results in those examples discussed here. The method given here is somewhat simpler and easier to program than the method given by Moore (1961) or a method due to the writer (Day, 1963) for the less general equation $u_{xy} = f(x, y, u)$.

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Table 1
Errors*

$x \backslash y$	1·0	2·0	3·0	4·0
1·0	7·29	9·70	6·49	4·03
2·0	9·70	29·45	34·11	25·87
3·0	6·49	34·11	80·38	95·07
4·0	4·03	25·87	95·07	211·36
4·5	3·37	22·24	88·90	246·82

* All errors in the table are multiplied by 10^{-5}.

Table 2
Errors*

$x \backslash y$	1·2	1·4	1·6	1·8	2·0
1·2	5·46	20·08	38·18	53·56	64·95
1·4	11·43	44·27	81·72	114·91	139·41
1·6	17·70	69·30	128·16	180·20	218·37
1·8	24·22	95·57	176·94	248·67	300·96
2·0	30·95	122·88	227·61	319·68	386·44

* All errors in the table are multiplied by 10^{-5}.

Table 3
Errors*

$x \backslash y$	1·0	2·0	3·0	4·0
1·0	20·37	31·54	37·84	41·47
2·0	31·54	51·68	64·81	73·44
3·0	37·84	64·87	84·28	98·31
4·0	41·47	73·44	98·31	117·55

* All errors in the table are multiplied by 10^{-5}.

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