The numerical solution of sequential decision problems involving parabolic equations with moving boundaries

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Numerical methods are described for finding the boundary separating the continuation and stopping regions for a class of sequential decision problems whose minimal cost function satisfies a parabolic partial differential equation. The problem is shown to be similar to that of finding the moving boundary in Stefan problems, and techniques used in the solution of Stefan problems are modified for use in decision problems. Numerical results for a particular sequential decision problem are given.

1. Introduction

When testing a hypothesis concerning the parameter of a statistical distribution it is often possible to devise an experiment in which observations are made in succession. After each observation a decision can then be made to accept the hypothesis, to reject the hypothesis, or to carry out a further observation. Corresponding to these decisions, a graph of the observations can be divided into three regions. Provided observations lie in the central *continuation region*, the experiment is continued. However, if an observation falls in one of the outer *stopping regions* the experiment is discontinued and the appropriate conclusion drawn regarding the hypothesis.

In some sequential decision problems of this kind, the decision to stop or continue can be based on a comparison between the cost of making an immediate decision which is possibly incorrect, and the cost of taking further observations in the hope of reducing the chance of an incorrect decision. Bather (1962) discusses several examples of sequential decision problems in which the minimal expected cost function satisfies a parabolic partial differential equation. The continuation and stopping regions are separated by curved boundaries on which various types of boundary condition are given. Explicit solutions of these problems are not usually available but Bather develops analytical techniques for finding inner and outer approximations to the boundaries.

The purpose of the present paper is to show that the solution of sequential decision problems of this type can be found by the use of numerical techniques. In order to illustrate the techniques in detail and provide comparative results, the specific parabolic equation

$$\frac{\partial^2 f}{\partial x^2} + x\frac{\partial f}{\partial x} + 2t\left(c + \frac{\partial f}{\partial t}\right) = 0, 0 \leqslant x < x(t), t > 0 \quad (1.1)$$

together with the boundary conditions

$$f(x(t),t) = \phi(x(t)), t > 0 \quad (1.2)$$

$$\frac{\partial f}{\partial x}(x(t), t) = \phi'(x(t)), t > 0 \quad (1.3)$$

$$\frac{\partial f}{\partial x}(0, t) = 0, t > 0 \quad (1.4)$$

where $\phi(x) = \int_x^\infty (2\pi)^{-1/2} \exp(-\tfrac{1}{2}y^2)dy$

will be investigated. However, unless a statement is made to the contrary, the methods described are applicable with only slight modification to more general sequential decision problems.

Since $\phi''(x) + x\phi'(x) = 0$, the function $u(x, t) = f(x, t) - \phi(x)$ satisfies the same form of parabolic equation as $f(x, t)$, namely

$$\frac{\partial^2 u}{\partial x^2} + x\frac{\partial u}{\partial x} + 2t\left(c + \frac{\partial u}{\partial t}\right) = 0, 0 < x < x(t), t > 0 \quad (1.5)$$

but with the modified boundary conditions

$$u(x(t), t) = 0, t > 0 \quad (1.6)$$

$$\frac{\partial u}{\partial x}(x(t), t) = 0, t > 0 \quad (1.7)$$

$$\frac{\partial u}{\partial x}(0, t) = (2\pi)^{-1/2}, t > 0. \quad (1.8)$$

This formulation shows clearly the similarity of sequential decision problems to the group of moving boundary problems generally referred to as Stefan problems (see, for example, Douglas (1961), p. 46).

Numerical methods which have been developed for Stefan problems and which can be modified for use in decision problems are described briefly in Section 2. In Section 3 a modification of a discretization due to Douglas and Gallie (1955) is described in more detail. This can be combined with the use of Lanczos τ-methods developed by Wragg (1966) to give an efficient means of determining the boundary $x(t)$ from the system (1.5), . . ., (1.8). Numerical results are given in Section 4.

2. Numerical methods of solution

A number of finite-difference schemes have been proposed and used in the numerical solution of Stefan problems. Trench (1959) used a fixed time step k and

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space step h and derived a simple scheme giving the temperature distribution explicitly for each successive time step. A method described by Crank (1957) involving Lagrange interpolation formulae for points near the boundary is somewhat similar. Both methods, however, suffer from the usual stability condition associated with the simple explicit finite-difference method of solving the heat conduction equation, $k \leqslant \frac{1}{2}h^2$. Ehrlich (1958) proposed a more complicated method, using fixed space and time steps, based on the Crank–Nicolson method (1947); the numerical evidence presented indicated that this gave satisfactory convergence to the true solution.

Evans, Isaacson and MacDonald (1950) have developed an alternative approach to the solution of Stefan problems in which the boundary is found as the solution of an integral equation, and the temperature distribution is given by an integral involving the boundary. Using Laplace transforms on the time variable, an ordinary second–order differential equation in the space variable is obtained which is solved analytically. Inverse transformation then yields the integral for the temperature distribution from which the integral equation for the boundary is derived as a consequence of the boundary conditions.

Sack (1965) has used Fourier transforms on the space variable to transform the decision problem (1.5), . . ., (1.8) into the integral equation

$$\exp\left(-\tfrac{1}{2}x^2\right) = 2c\int_t^\infty \frac{d\theta}{(1-t/\theta)^{1/2}}\exp\left\{-\frac{x^2+tX^2(\theta)/\theta}{2(1-t/\theta)}\right\}\sinh\left\{\frac{x(t/\theta)^{1/2}X(\theta)}{(1-t/\theta)}\right\},\ x \geqslant X(t). \quad (2.1)$$

In particular, putting $x = X(t)$ in (2.1) yields a non-linear integral equation for the boundary of the decision problem. Equation (2.1) can be solved numerically by means of simple quadrature and iteration using

$$X(t) \sim (8\pi)^{-1/2}/(ct) \quad (2.2)$$

to approximate $X(t)$ for large t and stepping backwards in time. This approach does not seem as versatile as the methods described in Section 3, but the integral equation can be of considerable use in analytical investigations since it provides a means of obtaining asymptotic expansions for the boundary.

A more useful numerical method is due to Crank (1957) which involves transforming the problem so that the moving boundary becomes fixed. Applying this method to the system (1.5), . . ., (1.8) the singularity in $\partial u/\partial x$ at infinity can first be removed by the transformation

$$\xi = xt,\ \tau = 1/t \quad (2.3)$$

and then the boundary can be fixed by the transformation

$$\eta = \xi/\xi_1 \quad (2.4)$$

where $\xi_1 = tX(t)$. Finite-difference methods can then be used to solve the transformed problem.

3. Combined Douglas–Gallie and Lanczos τ-methods

The technique described by Douglas and Gallie (1955) can be combined with the use of Lanczos τ-methods described by Wragg (1966) to provide an efficient and flexible method of solving decision problems.

In the Douglas–Gallie method the step in space is kept fixed but the time step is varied so that for each step in time the boundary moves through precisely one space interval. That is, equal intervals Δx are taken in the x-direction and variable intervals Δt_i in the t-direction. With this discretization, $U_{i,n}$ denotes an estimate of $u(x, t)$ at the mesh point

$$x = x_i = i\Delta x,\ t = t_n = \sum_{i=0}^{n-1}\Delta t_i.$$

Since Δt_0 is infinite, numerical calculations are not possible unless some modification is introduced to eliminate Δt_0. However, assuming that $U_{0,0} = 0$, formal finite-difference representations of (1.5), . . ., (1.8) for the interval Δt_0 imply that

$$U_{1,1} = 0,\ U_{0,1} = 0,\ U_{-1,1} = -(2\pi)^{-1/2}\Delta x,$$
$$t_1 = (8\pi)^{-1/2}/(c\Delta x) \quad (3.1)$$

the value of t_1 agreeing with the asymptotic behaviour of the boundary derived theoretically by Bather (1962), p. 608.

In practice, therefore, for a given value of Δx the values in (3.1) are assumed as the starting point for numerical work, that is $U_{i,n}$ is taken as an approximation to $u(x, t)$ at the point

$$x = x_i = i\Delta x,\ t = t_n = \sum_{i=1}^{n-1}\Delta t_i.$$

Thus at a general value t_n,

$$t_n = t_{n-1} - \Delta t_{n-1} \quad (3.2)$$

and the finite-difference representations of (1.6), (1.7), (1.5) and (1.8) become

$$U_{n,n} = 0 \quad (3.3)$$

$$\frac{U_{n,n} - U_{n-1,n}}{\Delta x} = 0 \quad (3.4)$$

$$\frac{U_{i,n} - 2U_{i-1,n} + U_{i-2,n}}{(\Delta x)^2} + (i-1)\Delta x\frac{U_{i,n} - U_{i-2,n}}{2\Delta x} + 2t_n\left[c + \frac{U_{i-1,n-1} - U_{i-1,n}}{\Delta t_{n-1}}\right] = 0 \quad (3.5)$$

for $i = n, n-1, \ldots, 1$, and

$$\frac{U_{0,n} - U_{-1,n}}{\Delta x} = (2\pi)^{-1/2}. \qquad (3.6)$$

Equations (3.3), . . ., (3.6) are solved by iteration. A value of Δt_{n-1} is assumed, t_n is evaluated from (3.2), the $U_{i,n}$ are evaluated by recursion from (3.3), (3.4), (3.5), and the test function

$$U_{-1,n} - U_{0,n} + (2\pi)^{-1/2}\Delta x \qquad (3.7)$$

is evaluated. The value of Δt_{n-1} is then modified, using an inverse interpolation procedure, until the value of the test function becomes sufficiently small.

This process is used to provide starting values so that the determination of the boundary can be continued by the use of Lanczos τ-methods which, for small intervals, are more efficient in terms of storage space and computer time (see Wragg, 1966).

The application of Lanczos τ-methods to the system (1.5), . . ., (1.8) is very similar to their use in the Stefan problem. It is sufficient here to note that the canonical polynomials $Q_m(x)$ for the differential operator

$$\frac{d^2}{dx^2} + x\frac{d}{dx} + k$$

where $k = -2t_1/\Delta t$, are given by

$$Q_m(x) = \sum_{s=0}^{m} q_{m,s}x^s$$

where

$$q_{m,s} = \begin{cases} 0, \text{ if } m < s \text{ or if } (m+s) \text{ is odd} \\ (-1)^{\frac{3m+s}{2}} \cdot \dfrac{m!}{s!} \cdot \dfrac{g(s-2)}{g(m)} \text{ otherwise.} \end{cases}$$

The function $g(r)$ is defined by $g(r) = 1$ for $r < 0$, $g(0) = k$, $g(1) = 1 + k$ and $g(r) = (r + k)g(r - 2)$ for $r \geqslant 2$.

4. Numerical results

All the methods described in Section 2 have been used to obtain numerical solutions of (1.5), . . ., (1.8) with $c = (8\pi)^{-1/2}$. For those methods which have stability restrictions, the calculations were restricted to determination of the boundary over a small time interval. The boundary has been determined, however, over the whole range of t using the modified method of Douglas and Gallie (1955), the solution of the integral equation (2.1), the method of Crank (1957) involving transformation to a fixed boundary, and the combined Douglas–Gallie and Lanczos τ-methods described in Section 3. Results from all these methods were in excellent agreement.

Typical results obtained by using the method of Section 3 are shown in **Table 1**. The modified Douglas–Gallie method has been used with $\Delta x = 0{\cdot}01$ to integrate to $x = 1{\cdot}0$ and then the Lanczos τ-method has been used with $\Delta t = 0{\cdot}01$ to continue the integration to $x = 1{\cdot}6$.

Table 1

Values of t corresponding to the tabulated values of x obtained by using a combination of the Douglas–Gallie and τ-methods with $\Delta x = \Delta t = 0{\cdot}01$. Asterisks denote interpolated values

$x = 0{\cdot}1$	9·929
$x = 0{\cdot}2$	4·854
$x = 0{\cdot}3$	3·124
$x = 0{\cdot}4$	2·238
$x = 0{\cdot}5$	1·696
$x = 0{\cdot}6$	1·329
$x = 0{\cdot}7$	1·065
$x = 0{\cdot}8$	0·866
$x = 0{\cdot}9$	0·712
$x = 1{\cdot}0$	0·590
$x = 1{\cdot}1$	0·487*
$x = 1{\cdot}2$	0·409*
$x = 1{\cdot}3$	0·343*
$x = 1{\cdot}4$	0·289*
$x = 1{\cdot}5$	0·244*
$x = 1{\cdot}6$	0·205*

Table 2

Boundary values of x corresponding to the tabulated values of t obtained by using (b) a combination of the Douglas–Gallie and τ-methods with $\Delta x = \Delta t = 0{\cdot}01$, compared with (a) an outer boundary and (c) an inner boundary evaluated by Bather (1962). Asterisks denote interpolated values

	(a)	(b)	(c)
$t = 0{\cdot}1$	2·41	2·02*	1·33
$t = 0{\cdot}2$	2·03	1·62*	1·07
$t = 0{\cdot}5$	1·39	1·09*	0·74
$t = 1{\cdot}0$	0·87	0·73*	0·52
$t = 1{\cdot}5$	0·62	0·55*	0·40
$t = 2{\cdot}0$	0·48	0·44*	0·33
$t = 2{\cdot}5$	0·39	0·36*	0·28

Finally, **Table 2** compares boundary values obtained by using the method of Section 3 with upper and lower limits of the boundary calculated by Bather (1962), p. 613.

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Book Review

Pattern Recognition, by Leonard Uhr, 1966; 393 pages. (London and New York: *John Wiley and Sons Ltd.*, 68s. cloth, 45s. paper.)

This is a most stimulating book. Leonard Uhr, who is well known for his work on computer programmed models of visual perception and cognition, has collected together from a range of disciplines important papers relevant to pattern recognition in all its aspects. He attempts "to focus on the problem of pattern recognition as it would be posed by someone interested in the psychological functions of perception and cognition". The result is a gold-mine of information and ideas.

There are five sections each of which contains approximately five papers drawn usually from the last fifteen years, but occasionally from considerably earlier. The first section is particularly pleasing for it introduces us to the subject by way of the thinking of such men as Peirce, Cassirer and Wittgenstein. Thus we are able to decide for ourselves the value of the more philosophical and theoretical approaches to the subject.

The second section is devoted to some of the experimental evidence on visual perception. Here are an important survey and discussion by Vernon of the nature of perception, and an early attempt by Attneave and Arnoult to study quantitatively the concept of shape—work which pioneered the computer scientist's now established approaches. Section 3 deals with attempts to interpret the experimental evidence. Deutsch's model of shape recognition in terms of a two-dimensional network of cells is accompanied by a paper by Dodwell discussing theories of discrimination learning with special reference to shape discrimination. Here also is a paper by Reichardt who is remarkably successful in modelling aspects of visual perception in the beetle Chlorophanus.

Section 4 brings us to neurophysiological results which are directly relevant to models of perception and their computer implementation. Experiments to investigate the organization and behaviour of retinal cells in the frog are described in a paper by Barlow, and important results revealing how the excitation of a specific visual cortex cell in the cat may be determined by a complex but precise stimulus are described in a paper by Hubel and Wiesel. A paper by Young presents a neurophysiological model of shape perception in the octopus and discusses the mechanics of motivation and reward.

Finally we reach the editor's own field, that of attempts to create digital computer systems of perception and cognition comparable with those we observe in nature. The approach by way of adaptive networks is represented by Roberts' extension of Rosenblatt's "Perceptron". More structured models described include Selfridge's "Pandemonium", and Uhr and Vossler's program that decides for itself what features to look for in the unknown pattern. The section, and the book, ends with a recent paper by Uhr himself in which he surveys the present and future of pattern recognition programs.

Uhr has made a rather personal selection of papers, but I do not at all regret this. His own views are sufficiently ordered to bind together material from many sources, and, even where he is perhaps being unorthodox, as in his willingness to accept introspection as a valid source of information, he is almost always convincing. Thus I find this book an excellent introduction to, and brief survey of, the work of the psychologist of perception and the computer scientist trying to program feature extraction and pattern recognition. Although the latest work in the computer area, for example that of Kamentsky and Liu, and Marrill's Cyclops project, is not included, Uhr in his preface makes clear to where the reader should next turn.

The book is equipped with subject and name indexes, and the bibliographies are often very extensive. It is a pity that so desirable a work is marred, in the reviewer's copy at least, by printing which has occasionally produced faded or blurred pages. But this is a small matter.

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