Some computational results of an improved A.D.I. method for the Dirichlet problem

By G. Fairweather and A. R. Mitchell*

The alternating direction implicit (A.D.I.) method of the present authors (Mitchell and Fairweather, 1964) is modified and used to solve the nine-point Laplace difference equation in a square. The greater accuracy of the method, h^4 as compared with h^2 for the Peaceman–Rachford method, where h is the mesh size, is achieved with no appreciable increase in time or effort.

The new A.D.I. method is then combined with the Schwarz Alternating Procedure to solve the nine-point difference equation in non-rectangular regions with sides parallel to the co-ordinate axes. Special attention is given to the re-entrant L-shaped region.

1. Introduction

Consider Laplace's equation in two-space variables

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad (1)$$

subject to the boundary condition $u(x, y) = f(x, y)$ for (x, y) a point on the boundary of the unit square Ω, $0 \leqslant x, y \leqslant 1$. If a uniform mesh of length h in each co-ordinate direction is imposed on Ω, then (1) may be approximated at an internal node of Ω by

$$[-(\delta\hat{x}^2 + \delta\hat{y}^2) + f\delta\hat{x}^2\delta\hat{y}^2]u(x, y) = 0, \qquad (2)$$

where f is a parameter, and $\delta\hat{x}$, $\delta\hat{y}$ are the central-difference operators in the x, y directions, respectively. When $f = 0$ and $-\frac{1}{6}$, (2) yields the well-known five-point and nine-point difference approximations of (1) which are accurate to order h^2 and h^4, respectively. If $Nh = 1$, the totality of equations (2) gives rise to $(N - 1)^2$ linear equations in $(N - 1)^2$ unknowns of the form

$$\boldsymbol{Bu} = \boldsymbol{g} \qquad (3)$$

where the matrix B may be written in the form

$$B = H + V + fHV$$

and $\boldsymbol{g}$ is a vector of order $(N - 1)^2$ arising from the boundary values of the problem. The matrices H and V are such that if $[H\boldsymbol{u}](x, y)$ denotes the component of the vector $H\boldsymbol{u}$ corresponding to the internal node (x, y) of Ω, and similarly for $[V\boldsymbol{u}](x, y)$, then

$$[H\boldsymbol{u}](x, y) \equiv -u(x - h, y) + 2u(x, y) - u(x + h, y)$$
$$[V\boldsymbol{u}](x, y) \equiv -u(x, y - h) + 2u(x, y) - u(x, y + h),$$

where the points $(x \pm h, y \pm h)$ are internal mesh points of the region.

The present authors (Mitchell and Fairweather, 1964) formulated an alternating direction implicit (A.D.I.) method for solving (3) which may be written in the form

$$[I + \tfrac{1}{2}(r + f)H]\boldsymbol{u}^{(m+1/2)} = [I - \tfrac{1}{2}(r - f)V]\boldsymbol{u}^{(m)} + b_1\boldsymbol{g}$$
$$[I + \tfrac{1}{2}(r + f)V]\boldsymbol{u}^{(m+1)} = [I - \tfrac{1}{2}(r - f)H]\boldsymbol{u}^{(m+1/2)} + b_2\boldsymbol{g} \qquad (4)$$

where $\boldsymbol{u}^{(m)}$, $\boldsymbol{u}^{(m+1)}$ are the mth and $(m + 1)$th estimates of $\boldsymbol{u}$, the solution vector, $\boldsymbol{u}^{(m+1/2)}$ is treated as an auxiliary vector which is not retained from one complete iteration to the next, r is a parameter, and b_1, b_2 are scalars to be determined. The iteration procedure described by (4) converges for all r if $f \geqslant \left(-2\cos^2\frac{\pi}{2N}\right)^{-1}$, and so $\boldsymbol{u}^{(m)} = \boldsymbol{u}^{(m+1)} = \boldsymbol{u}$ for m sufficiently large.

In order to find b_1 and b_2, we eliminate $\boldsymbol{u}^{(m+1/2)}$ from (4) and set $\boldsymbol{u}^{(m+1)} = \boldsymbol{u}^{(m)} = \boldsymbol{u}$. This gives

$$[H + V + fHV]\boldsymbol{u} = \frac{1}{r}[(b_1 + b_2)I + \tfrac{1}{2}\{(r + f)b_2 - (r - f)b_1\}H]\boldsymbol{g},$$

which reduces to (3) if

$$b_1 = \tfrac{1}{2}(r + f), \quad b_2 = \tfrac{1}{2}(r - f).$$

Thus the A.D.I. method for solving (2) in a square is

$$\left.\begin{aligned}[I + \tfrac{1}{2}(r + f)H]\boldsymbol{u}^{(m+1/2)} &= [I - \tfrac{1}{2}(r - f)V]\boldsymbol{u}^{(m)} + \tfrac{1}{2}(r + f)\boldsymbol{g}\\ [I + \tfrac{1}{2}(r + f)V]\boldsymbol{u}^{(m+1)} &= [I - \tfrac{1}{2}(r - f)H]\boldsymbol{u}^{(m+1/2)} + \tfrac{1}{2}(r - f)\boldsymbol{g}.\end{aligned}\right\} \qquad (5)$$

It can easily be shown that each step of the method given by (5) is consistent only if $f = 0$, when it reduces to the Peaceman–Rachford method (1955). Because of this lack of consistency for values of f other than $f = 0$, the A.D.I. method defined by (5) cannot be obtained from the general formulation of A.D.I. methods given recently by Douglas and Gunn (1964), unless, of course, $f = 0$.

It is shown by the present authors (1964) that provided r is kept constant during the iterations, the best con-

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vergence rate for a given value of $N\left(\equiv \frac{1}{h}\right)$ and f is given by

$$\frac{1 + f\sin^2\frac{\pi}{N} - \sin\frac{\pi}{N}\left(1 + 2f + f^2\sin^2\frac{\pi}{N}\right)^{1/2}}{1 + f\sin^2\frac{\pi}{N} + \sin\frac{\pi}{N}\left(1 + 2f + f^2\sin^2\frac{\pi}{N}\right)^{1/2}}, \qquad (6)$$

and the associated value of r necessary to yield the optimum convergence is

$$r^* = \left(1 + 2f + f^2\sin^2\frac{\pi}{N}\right)^{1/2}\sin\frac{\pi}{N}. \qquad (7)$$

In fact, if the A.D.I. method (5) is used to solve (1), it appears that a balance is required between the rate of convergence (an optimum when f is large and positive) and the accuracy of the difference formula (an optimum when $f = -\frac{1}{6}$). It will now be shown by means of numerical calculations that, when $f = -\frac{1}{6}$, the improvement in accuracy more than compensates for the slower convergence, particularly in the case of variable iteration parameters. In fact, (5) with $f = -\frac{1}{6}$ is a better method for solving the model problem of (1) in a square than the Peaceman–Rachford method, which is generally recognized as being the best of the existing methods (Birkhoff, Varga and Young, 1962). The problem which is used to illustrate this consists of equation (1) together with the boundary conditions

$$u(0, y) = u(1, y) = 0, \quad 0 \leqslant y \leqslant 1 \quad : \quad u(x, 0) = u(x, 1) = \sin \pi x, \quad 0 \leqslant x \leqslant 1. \qquad (8)$$

The theoretical solution of this problem is

$$u(x, y) = \operatorname{sech}\frac{\pi}{2}\cosh \pi(y - \tfrac{1}{2})\sin \pi x. \qquad (9)$$

Each experiment is started with $u^{(0)}(x, y) = 0$, for all (x, y) inside the unit square.

2. Variable iteration parameters

If the iteration parameter r in (5) is allowed to vary and take the value $r_i (1 \leqslant i \leqslant m)$ for each of m successive iterations, then it can easily be shown following Birkhoff, Varga and Young (1962), that the optimum parameters for the A.D.I. method (5) are those which minimize

$$\max_{a \leqslant \gamma \leqslant b} \prod_{i=1}^{m} \frac{r_i - \gamma}{r_i + \gamma}, \qquad (10)$$

where

$$a = f + \frac{1}{2\cos^2\frac{\pi}{2N}} \quad \text{and} \quad b = f + \frac{1}{2\sin^2\frac{\pi}{2N}}.$$

Several authors have obtained parameter sequences which are simple to use and which approximately minimize (10) for the Dirichlet problem in a square. One of the most satisfactory is that presented by Wachspress (1957) who obtains the parameter sequence

$$r_i = a\left(\frac{b}{a}\right)^{\frac{i-1}{m-1}}, \quad (i = 1, 2, \ldots, m) \qquad (11)$$

where m is the smallest integer such that

$$\delta^{2(m-1)} \leqslant a/b, \qquad (12)$$

where $\delta = \sqrt{2} - 1$. This is the parameter sequence used in the present paper and we refer to the values of r_i given by (11) as the *Wachspress parameters.*

It is shown by Birkhoff, Varga and Young (1962) that in general the Peaceman–Rachford method with Wachspress parameters is superior to all variants of the method of successive over-relaxation for the numerical solution of the Dirichlet problem. This is particularly so when N is large. We now compare the Peaceman–Rachford method which is (5) with $f = 0$, with the optimum A.D.I. method given by (5) with $f = -\frac{1}{6}$, using Wachspress parameters in both cases.

The number of parameters required for a given value of N is calculated from (12) both for $f = 0$ and $f = -\frac{1}{6}$. The results are shown in **Table 1.** There is no significant difference in the number of parameters required in the two cases. For the values of N quoted, only $N = 20$ and $N = 50$ require one more parameter in the optimum case than in the Peaceman–Rachford case. The values of the Wachspress parameters themselves are shown in **Table 2** for $N = 10$, 20 and $f = 0, -\frac{1}{6}$.

Table 1

No. of Wachspress parameters

N \ f	0	$-\frac{1}{6}$
10	4	4
20	4	5
40	5	5
50	5	5
100	6	6
500	8	8
1000	9	9

Calculations are carried out for $f = 0$ and $-\frac{1}{6}$ with $N = 10$ and 20, and one sequence of parameters is required for convergence in each case. Each calculation is continued until the error, that is, the difference between the theoretical and the computed solutions of (1) settles down in the seventh decimal place. The errors at the centre node of the square where the theoretical solution is 0·398,536,8 are shown in **Table 3,** and the improved accuracy of the A.D.I. method with $f = -\frac{1}{6}$ is demonstrated. In general, the number of iterations required for convergence is the same for the optimum method as for the Peaceman–Rachford method.

Table 2

Wachspress iteration parameters

N \ f	0	$-\frac{1}{6}$
10	0·512,542,815,5 1·750,874,996,0 5·981,087,158,5 20·431,729,094,5	0·345,876,148,8 1·343,372,308,4 5·217,616,668,0 20·265,062,427,9
20	0·503,096,979,3 2·739,443,045,7 14·916,702,960,1 81·223,819,398,8	0·336,430,312,7 1·325,467,780,9 5·222,076,525,4 20·573,931,430,2 81·057,152,732,1

In addition, the relative amounts of arithmetic per iteration are comparable in the two methods, and so the considerable improvement in accuracy is achieved without additional machine time.

3. Extension of method to more general regions

The A.D.I. method given by (5) applies only when the region under consideration is rectangular. We now investigate the possibility of using (5) with $f = -\frac{1}{6}$ on non-rectangular regions, bearing in mind that a nine-point difference approximation of (1) can only be employed when the boundaries of the region are parallel to the co-ordinate axes. Except for the square or rectangle, however, regions with boundaries parallel to the axes do not lend themselves directly to solution by the A.D.I. method given by (5) with $f \neq 0$. This will be illustrated by means of a simple example.

Suppose it is required to solve Laplace's equation in the region shown in **Fig. 1.** It is required to find the value of the function at the circled numbered mesh points. If the nine-point formula (2) with $f = -\frac{1}{6}$ is used, we obtain the set of equations (3) where

$$B = \frac{1}{6}\begin{bmatrix} -20 & 4 & 4 \\ 4 & -20 & 1 \\ 4 & 1 & -20 \end{bmatrix} \quad \text{and} \quad u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}.$$

However, for this problem,

$$H + V - \tfrac{1}{6}HV = \frac{1}{6}\begin{bmatrix} -20 & 4 & 4 \\ 4 & -20 & 1 \\ 4 & 0 & -20 \end{bmatrix} \neq B,$$

and so the A.D.I. method (5) with $f = -\frac{1}{6}$ will not solve the required set of equations.

However, the A.D.I. method given by (5) can be used in regions with boundaries parallel to the axes, in conjunction with the numerical alternating procedure of Miller (1965), which is a numerical analogue of the Schwarz alternating procedure (Kantorovich and Krylov, 1958). This procedure enables one to solve the Dirichlet problem for Laplace's equation on the union of two overlapping plane regions, provided the Dirichlet problem is solvable on each separately, and that the boundaries of the regions intersect at non-zero angles.

Table 3

N \ f	0	$-\frac{1}{6}$
10	−0·004,687,2	−0·000,000,1
20	−0·001,178,3	−0·000,000,0

Table 4

N \ f	0	$-\frac{1}{6}$
10	−0·003,241,3	−0·000,000,1
20	−0·000,824,3	−0·000,000,0

To illustrate how this numerical alternating procedure can extend the use of the A.D.I. method (5) to more general regions, we consider the solution of Laplace's equation in the L-shaped region, illustrated in **Fig. 2,** which consists of the unit square with a square of area 4/25 removed from one corner. There is no special significance in this region and in fact any L-shaped region can be considered. For convenience, the boundary values at nodes on AB, BC, CD and FA are taken from (8), and those at nodes on DE and EF from (9). Thus the theoretical solution of the problem is given by (9). Calculations are again carried out for $N = 10$ and 20.

This problem is solved first by direct application of the Peaceman–Rachford method to the L-shaped region using Wachspress parameters. Although the theory on which the determination of these parameters is based does not apply in this case, since the matrices corresponding to H and V no longer commute, it has been shown by Young and Ehrlich (1960) and Price and

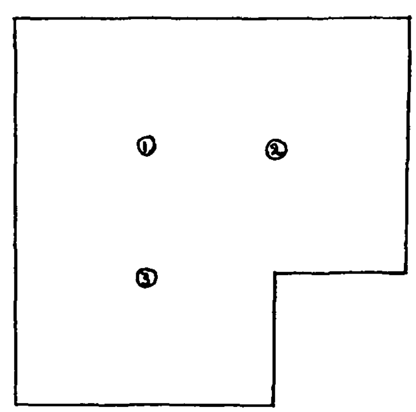

Fig. 1

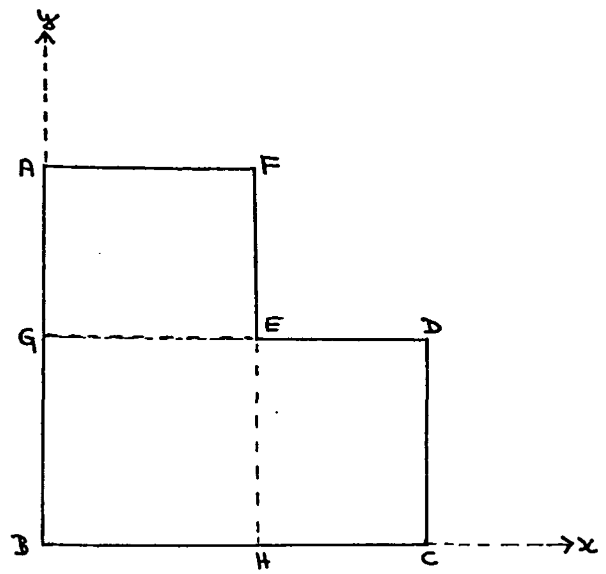

Fig. 2

Varga (1962) that their use produces reasonably rapid convergence of the iterative procedure. The parameters are given in Table 2 for $N = 10$, 20 and $f = 0$, and the Peaceman–Rachford method converges after six sequences of these parameters in each case. The maximum errors, which occur at the node $x = 1/2$, $y = 3/10$ where the theoretical solution is $0 \cdot 479{,}827{,}2$, are given in **Table 4.**

The problem is next solved using the A.D.I. method (5) with $f = -\frac{1}{6}$ together with the Schwarz alternating procedure. The L-shaped region is divided into two overlapping rectangles BCDG (region R_1) and ABHF (region R_2) and, initially we place $u^{(0)}(x, y) = 0$, for all (x, y) inside the region. By means of (5) with $f = -\frac{1}{6}$ and one sequence of the appropriate Wachspress parameters, a solution is obtained in R_1 with the values of u along GE equal to zero. This calculation gives a first estimate of the function u along EH which enables a solution to be obtained in R_2, again using (5) with $f = -\frac{1}{6}$ and one sequence of the Wachspress parameters. A new estimate of the values along GE is thus obtained. This procedure is continued until

$$|u^{(m+1)}(x, y) - u^{(m)}(x, y)| < 10^{-7}$$

for all (x, y) inside the region. The maximum errors after seven applications of the alternating procedure are given in Table 4, again at the node $x = 1/2$, $y = 3/10$.

In view of the fact that the convergence of the overall procedure depends on the convergence of the A.D.I. method together with the convergence of the alternating procedure, the latter depending only on the geometry of the region (Miller, 1965), it is difficult to give a meaningful estimate of the rate of convergence for the iterative method for overlapping regions.

In principle, there is no limit to the number of overlapping regions to which the alternating procedure can be extended, and so theoretically the A.D.I. method can be used to solve Laplace's equation to h^4 accuracy in any region bounded by lines parallel to the co-ordinate axes, provided no singularity in u or its derivatives occurs anywhere in the region. In practice, however, the method becomes rather tedious if a large number of overlapping regions occurs. It should be noted that Saul'ev (1963) has also devised a technique for using A.D.I. methods in non-rectangular regions. This method involves the solution of a related problem in the smallest rectangle which encloses the original region.

4. Singularities in the L-shaped region

In the last section, a simple example was chosen to illustrate how the Schwarz alternating procedure can be used in conjunction with the A.D.I. method of (5) to solve the Dirichlet problem in an L-shaped region, and h^4 accuracy was obtained uniformly over the field. However, in many physical problems, the boundary conditions are such that the function vanishes on the straight boundaries meeting at the re-entrant corner, and so the solution of Laplace's equation in plane polar coordinates can be written in the form

$$u = \sum_{n=1}^{\infty} b_n r^{\left(\frac{2}{3}\right)n} \sin \tfrac{2}{3} n\theta$$

where the origin of the coordinate system is taken at the corner [see Fox (1962), p. 303]. Accordingly, there is a discontinuity in some derivative at the re-entrant corner, and any finite-difference method will lose accuracy in the vicinity of the corner. This point was discussed to some extent by Young (1955) who used successive over-relaxation methods to solve five- and nine-point replacements of Laplace's equation in an L-shaped region, and showed that for a prescribed set of boundary conditions the solution obtained from the nine-point replacement was considerably more accurate than that from the five-point formula even in the vicinity of the re-entrant corner. As the theoretical solution of the problem considered by Young is unobtainable, the nature of the singularity at the re-entrant corner is unknown.

In order to discuss this point further, we consider three model problems in classical hydrodynamics where singularities occur at the re-entrant corner, $(r = 0)$. The problems consist of:

(1) flow round a right-angled bend where the stream function u is given by

$$u = r^{2/3} \sin \frac{2}{3} \theta$$

and singularities occur in first and higher derivatives with respect to r;

(2) flow past an infinite wedge with a $\pi/4$ semi-wedge angle, where

$$u = r^{4/3} \sin \frac{4}{3} \theta,$$

and the singularities occur in the second and higher derivatives;

Table 5

N \ f	0		$-\frac{1}{6}$	
10	(*a*) 0·016,885,3	(*b*) 0·000,261,6	(*a*) 0·011,254,2	(*b*) 0·000,250,3
20	(*a*) 0·006,666,6	(*b*) 0·000,113,8	(*a*) 0·004,418,4	(*b*) 0·000,092,0

Theoretical solutions: (*a*) 0·186,579,5 (*b*) 0·793,700,5

Table 6

N \ f	0		$-\frac{1}{6}$	
10	(*a*) 0·001,248,0	(*b*) 0·000,121,1	(*a*) 0·000,462,6	(*b*) 0·000,015,9
20	(*a*) 0·000,273,2	(*b*) 0·000,027,5	(*a*) 0·000,073,8	(*b*) 0·000,002,4

Theoretical solutions: (*a*) −0·040,197,3 (*b*) −0·287,647,6

Table 7

N \ f	0		$-\frac{1}{6}$	
10	(*a*) 0·000,170,8	(*b*) 0·000,045,9	(*a*) 0·000,006,6	(*b*) 0·000,000,1
20	(*a*) 0·000,040,8	(*b*) 0·000,011,6	(*a*) 0·000,000,3	(*b*) 0·000,000,0

Theoretical solutions: (*a*) 0·001,865,8 (*b*) —0·091,080,2

(3) flow past an infinite wedge with a $\pi/4$ semi-wedge angle where the stream line $u = 0$ has three branches at the corner. Here the stream function is given by

$$u = r^{8/3} \sin \frac{8}{3} \theta,$$

and the singularities are in the third and higher derivatives.

These problems are solved first using (5) with $f = 0$ and next using (5) with $f = -\frac{1}{6}$ together with the Schwarz alternating procedure. In each problem an L-shaped region is considered which has the right-angled corner or wedge tip as the re-entrant corner. The straight boundaries which intersect at the corner are the stream lines $u = 0$. The remaining boundary values of the stream function u are taken from the respective theoretical solutions.

For the three problems described, comparative errors in the two methods for $N = 10$, 20 are shown in **Tables 5, 6** and **7**, respectively. Results are quoted after six sequences of the Wachspress parameters for $f = 0$ and after six applications of the alternating procedure for $f = -\frac{1}{6}$, (*a*) at the node (1/2, 3/10) which is nearest to the corner and where the error is a maximum, and (*b*) at a typical point in the field away from the re-entrant corner. The nine-point formula ($f = -\frac{1}{6}$) is always the more accurate, particularly for problems (2) and (3). In the light of these results, it is probable that the problem considered by Young is similar to problem (2) or problem (3).

The present authors are indebted to D. E. Rutherford for pointing out that in problems arising in classical hydrodynamics involving an L-shaped region with $u = 0$ on the boundaries meeting at the re-entrant corner, this corner is either an infinite velocity point (problem (1)), or a stagnation point (problems (2) and (3)). In the latter case, the discontinuity is in the second or higher derivatives, and the nine-point formula, although losing accuracy, will be significantly more accurate than the five-point formula.

From the above evidence, it appears that the numerical solution of the Dirichlet problem in an L-shaped region is less prone to error than the numerical solution of the corresponding eigenvalue problem. For example, a recent account of methods for solving the latter problem (Reid and Walsh, 1965) shows that the nine-point formula gives a slightly poorer result than the five-point formula for the lowest eigenvalue.

As far as the present authors are aware, there have been only two previous attempts to solve the nine-point Laplace difference equation by means of an A.D.I. procedure. The first was by Samarskii and Andreev

(1963) who considered alternating direction methods for solving the iterative formula

$$(1 - r\delta^2\hat{x})(1 - r\delta\hat{y}^2)u^{(m+1)} = [(1 - r\delta x^2)(1 - r\delta\hat{y}^2) + r(\delta\hat{x}^2 + \delta\hat{y}^2 + \tfrac{1}{6}\delta\hat{x}^2\delta\hat{y}^2)]u^{(m)}.$$

It is not possible to factorize the right-hand side of this formula, and so it cannot be split into Peaceman–Rachford form like (5). As a result the examination of the convergence of the procedure is considerably more difficult and the method more complicated than the method of the present paper. The second attempt was by Cannon and Douglas (1964) who proposed a three-level alternating-direction iterative method. The presence of the extra level, of course, adds undue complication to the numerical procedure.

6. Acknowledgements

The calculations were carried out on the IBM 1620 computer of the University of St. Andrews. Dr. G. Fairweather's share of the work was performed whilst he was in receipt of a Carnegie Scholarship.

References

BIRKHOFF, G., VARGA, R. S., and YOUNG, D. M. (1962). *Alternating Direction Implicit Methods*, "Advances in Computers", Vol. 3, Acadamic Press, New York.

CANNON, J. R., and DOUGLAS, J. (1964). "Three-level Alternating Direction Iterative Methods", *Contributions to Differential Equations*, Vol. 3, pp. 189–198.

DOUGLAS, J., and GUNN, J. E. (1964). "A general formulation of Alternating Direction Methods, Part 1. Parabolic and Hyperbolic Equations", *Numerische Mathematik*, Vol. 6, pp. 428–453.

FOX, L. (1962). *Numerical Solution of Ordinary and Partial Differential Equations*, Pergamon Press, London.

KANTOROVICH, L. V., and KRYLOV, V. I. (1958). *Approximate Methods of Higher Analysis*, Interscience, New York.

MILLER, K. (1965). "Numerical Analogs to the Schwarz Alternating Procedure", *Numerische Mathematik*, Vol. 7, pp. 91–103.

MITCHELL, A. R., and FAIRWEATHER, G. (1964). "Improved Forms of the alternating direction methods of Douglas, Peaceman, and Rachford for solving parabolic and elliptic equations", *Numerische Mathematik*, Vol. 6, pp. 285–292.

PEACEMAN, D. W., and RACHFORD, H. H. (1955). "The numerical solution of parabolic and elliptic differential equations", *J. Soc. Indust. and Appl. Math.*, Vol. 3, pp. 28–41.

PRICE, H. S., and VARGA, R. S. (1962). "Recent numerical experiments comparing successive overrelaxation iterative methods with A.D.I. methods", Gulf Research and Development Co., Pittsburgh.

REID, J. K., and WALSH, J. E. (1965). "An elliptic eigenvalue problem for a re-entrant region", *J. Soc. Indust. and Appl. Math.*, Vol. 13, pp. 837–850.

SAMARSKII, A. A., and ANDREEV, V. B. (1963). "On a difference scheme of increased accuracy for equations of elliptic type with some spatial variables", *Z. Vycisl. Mat. i Mat. Fiz.*, Vol. 3, pp. 1006–1013.

SAUL'EV, V. K. (1963). "Solution of certain boundary value problems on high-speed computers by the fictitious-domain method", *Siburck. Mat. Z.*, Vol. 4, pp. 912–925.

WACHSPRESS, E. L. (1957). "A generalised two-space dimension multigroup coding for the IBM 704", C.U.R.E. report KAPL-1724, General Electric Co., New York.

YOUNG, D. M., and EHRLICH, L. (1960). "Numerical experiments involving boundary problems in Differential Equations", *Boundary Problems in Differential Equations*, pp. 143–162. University of Wisconsin Press, Madison.

YOUNG, D. M. (1955). "ORDVAC solutions of the Dirichlet problem", *J. Assoc. Comp. Mach.*, Vol. 2, pp. 137–161.

Book Review

Principles of Coding, Filtering and Information Theory, by Leonard S. Schwarz, 1963; 255 pages. (London: *Cleaver-Hume Press Ltd.*, 72s.)

Information Transmission, by Elwyn Edwards, 1964; 133 pages. (London: *Chapman and Hall Limited*, 15s.)

Here are two admirable books on the same subject, but written for very different people—Schwarz for the mathematical engineer and Edwards for the experimental psychologist. As this is not evident from the short titles, care should be taken to select the right book before ordering from a catalogue! Schwarz's *Principles of Coding, Filtering and Information Theory* covers modern statistical communication theory, coding, generalized harmonic analysis, signal detection and feedback communication, all treated in an elementary but fully professional manner. It is distinguished by exceptional clarity of expression, and every noteworthy aspect of the subject is introduced in the one convenient volume, which is well referenced.

Elwyn Edwards' *Information Transmission* is entirely different because it makes no assumption of mathematical literacy on the part of the reader. It comes as a shock to find that the experimental psychologist is thought to need an explanation of brackets and indices, but in thirteen pages the author gives all the mathematics he needs. (The definition of probability was demolished by Jeffreys long ago, but no matter.) It has always seemed to the reviewer that experimental psychologists cannot do anything very much with information theory except to use its definitions and terms. These provide him with something to plot. At the present stage, as at ten years ago, one can only hope that the concepts prove suggestive—clearly the author's hope also. He is to be congratulated on explaining the subject so simply and so readably.

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