of approximation with increasing m. These approximations can again be computed by using *MINSUMMOD* and *MINMAXMOD* without alteration.

Thus these algorithms provide adequate means of obtaining splines and similar approximations without recourse to special computing methods.

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Correspondence

To the Editor,
The Computer Journal.

Sir,

Papers by Parker and Crank (1964) and Keast and Mitchell (1966) have recently considered the stability of Crank and Nicolson's procedure (Crank and Nicolson, 1947) for solving the parabolic partial differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad (1)$$

with $u(x, 0) = f(x)$, $0 \leqslant x \leqslant 1$, and with boundary conditions

$$a_0 \frac{\partial u}{\partial x} + b_0 u = \lambda_0(t);\ x = 0,\ t > 0$$

$$a_1 \frac{\partial u}{\partial x} + b_1 u = \lambda_1(t);\ x = 1,\ t > 0.$$

Their results conceal what is an essentially simple situation. Consider the preparation of (1) for solution by a computer in the two following stages:

(*a*) The right-hand side of (1) is replaced by a suitable difference scheme in Δx, and the boundary conditions are incorporated to give (cf. Parker and Crank, 1964)

$$\dot{w} = \frac{1}{(\Delta x)^2}[-Uw + l];\ w(0) = c \qquad (2)$$

where $w(t)$ is a vector with $N + 1$ components approximating the value of $u(x, t)$ at $x = 0, \Delta x, 2\Delta x, \ldots, N\Delta x$. The physics of the problem can be a valuable guide at this stage: indeed it is safest to set up (2) directly from a discrete physical model (see Rosenbrock and Storey, 1965, pp. 8–15).

(*b*) The time derivatives in (2) are replaced by a difference scheme to give (cf. Parker and Crank, 1964)

$$v^{n+1} - v^n = r\{\theta[-Uv^{n+1} + l^{n+1}] + (1-\theta)[-Uv^n + l^n]\} \qquad (3)$$

$$[I + r\theta U]v^{n+1} = [I - r(1-\theta)U]v^n + k^n;\ v^0 = c \qquad (4)$$

where v^n approximates $w(n\Delta t)$ and $r = \Delta t/(\Delta x)^2$. So far as this stage is concerned we have the following simple result:

If (2) is stable (resp. asymptotically stable), and if $\frac{1}{2} \leqslant \theta \leqslant 1$, $r > 0$ [or if $0 \leqslant \theta < \frac{1}{2}$ and $0 < r \leqslant 1/(\frac{1}{2} - \theta)\lambda_{max}(U)$] then (4) is stable (resp. asymptotically stable).

Thus all the real difficulties regarding the stability of (4) are associated with stage (*a*), which belongs to the physical formulation of the problem rather than to Crank and Nicolson's procedure. Of course if (2) is unstable (or stable but not asymptotically stable) we have no right to expect (4) to be stable (or asymptotically stable).

To prove the result stated it is only necessary to write

$$c = \sum_{i=0}^{N} \alpha_i z_i \qquad (5)$$

where $$Uz_i - \lambda_i z_i = 0 \qquad (6)$$

(*Continued on p.* 324)

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```
    c[n − 1]: = b × d[n − 2];
    d[n − 1]: = t; d[n]: = c[n];
    c[n]: = a × t;
    c[n + 1]: = (m + n) × (2 × n − 1) × d[n]/((m − n) ×
        (2 × n + 1));
    comment this is the n.f. for the nth polynomial;
end
```

Editor's note

Material for this Supplement should be sent to the Algorithms Editor

P. Hammersley,
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London, E.C.1.

Correspondence (*continued from p.* 320)

This is always possible because U is similar to a symmetric matrix. The solution of (2) when $l = 0$ is

$$w(t) = \left[\exp\left(-\frac{1}{(\Delta x)^2}Ut\right)\right]c$$

$$= \sum_{i=0}^{N} \alpha_i z_i \exp\left(\frac{-\lambda_i t}{(\Delta x)^2}\right) \qquad (7)$$

or

$$w(n\Delta t) = \sum_{i=0}^{N} \alpha_i z_i [\exp(-r\lambda_i)]^n. \qquad (8)$$

On the other hand the solution of (4) with $k = 0$ is

$$v^n = \{[I + r\theta U]^{-1}[I - r(1-\theta)U]\}^n c \qquad (9)$$

$$= \sum_{i=0}^{N} \alpha_i z_i \left(\frac{1 - r(1-\theta)\lambda_i}{1 + r\theta\lambda_i}\right)^n. \qquad (10)$$

The replacement of (2) by (4) therefore replaces each factor $\exp(-r\lambda_i)$ in one time step of (8) by a factor

$$\frac{1 - r(1-\theta)\lambda_i}{1 + r\theta\lambda_i} \qquad (11)$$

whence the result follows.

This simple relationship between the stability properties of equations (2) and (4) does *not* necessarily persist when Crank and Nicolson's procedure with $\theta = \frac{1}{2}$ is applied to a non-linear or non-autonomous problem (Rosenbrock and Storey, 1965, pp. 173–175). Some formulae giving improved stability and truncation error have been suggested in an earlier note (Rosenbrock, 1963).

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Yours faithfully,

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16 June 1966.

To the Editor,
The Computer Journal.

Sir,

I should like to reply to the letter by K. Wright (this *Journal*, May 1966, p. 115) about my paper entitled "Error curves for Lanczos' 'selected points' method" (this *Journal*, January 1966, p. 372). I apologise for stating that Wright's statement about the form for the residual (this *Journal*, January 1964, p. 358) is incorrect. His letter clearly shows the source of my confusion.

However, I do not agree with the simpler derivation in the letter for the form of the residual. Although

$$r(x) = \acute{e}(x) - e(x)\frac{\partial F}{\partial y} \cdots$$

there is no justification in dropping the whole right-hand side except for the first term. The derivation given in my paper based on the Picard iteration does show how errors build up.

Incidentally, there are two typographical errors in my paper. In Table 4, H_{31} should read $-0{\cdot}089142227$ instead of $-0{\cdot}08142227$. In Table 5, G_{41} should read $0{\cdot}37699459$ instead of $0{\cdot}376994519$.

Yours sincerely,

W. KIZNER

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1 August 1966

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