Note on the numerical solution of integro-differential equations

By James Thomas Day*

A numerical method for the solution of integro-differential equations is devised. Two numerical examples are given.

1. In this paper a numerical method for the solution of integro-differential equations of the form

$$y'(x) = f(x, y(x)) + \int_{x_0}^{x} F(x, s, y(s))ds, \quad y(x_0) = y_0 \qquad (1.1)$$

is discussed. Two computational examples are given.

In the subsequent discussion $y(x_k)$ will denote the exact value of y at $x_k = x_0 + kh$. y_k will denote an approximate value of y at x_k. Likewise y'_k indicates an approximate value of $y'(x_k)$ at x_k.

2. To advance the step from $x = x_k$ to $x = x_k + h$, we first integrate (1.1) to obtain

$$\begin{aligned} y(x_k + h) &= y(x_k) + \int_{x_k}^{x_k+h} f(x, y(x))dx \\ &\quad + \int_{x_k}^{x_k+h} \int_{x_0}^{x} F(x, s, y(s))dsdx \qquad (2.1) \\ &= y(x_k) + I_1 + I_2. \end{aligned}$$

For I_1 we use the trapezoidal rule, together with the approximation $y_{k+1} = y_k + hy'_k$ for $y(x_{k+1})$ to yield

$$I_1 = h[f(x_k, y_k) + f(x_{k+1}, y_k + hy'_k)] + O(h^3). \qquad (2.2)$$

For I_2 we use the trapezoidal rule for the outer integral to obtain

$$I_2 = \frac{h}{2}\left[\int_{x_0}^{x_k} F(x_k, s, y(s))ds + \int_{x_0}^{x_{k+1}} F(x_{k+1}, s, y(s))ds\right] + O(h^3). \qquad (2.3)$$

We then again use the trapezoidal rule for the inner integral and again use the approximation $y_{k+1} = y_k + hy'_k$ to obtain

$$\begin{aligned} y_{k+1} = y_k &+ h[f(x_k, y_k) + f(x_{k+1}, y_k + hy'_k)]/2 \\ &+ h^2[F(x_k, x_0, y_0) + 2F(x_k, x_1, y_1) + \ldots \\ &+ 2F(x_k, x_{k-1}, y_{k-1}) + F(x_k, x_k, y_k)]/4 \\ &+ h^2[F(x_{k+1}, x_0, y_0) + 2F(x_{k+1}, x_1, y_1) + \ldots \\ &+ 2F(x_{k+1}, x_k, y_k) + \\ &F(x_{k+1}, x_{k+1}, y_k + hy'_k)]/4. \qquad (2.4) \end{aligned}$$

The derivative y'_{k+1} is calculated according to

$$\begin{aligned} y'_{k+1} &= f(x_{k+1}, y_{k+1}) + h[F(x_{k+1}, x_0, y_0) \\ &\quad + 2F(x_{k+1}, x_1, y_1) + \ldots + 2F(x_{k+1}, x_k, y_k) \\ &\quad + F(x_{k+1}, x_{k+1}, y_{k+1})]/2 \\ y'(x_0) &= f(x_0, y(x_0)). \end{aligned}$$

Examples

Our first computational example is the integro-differential equation

$$y' = 1 - \int_0^x y(s)ds$$

with solution $y(x) = \sin(x)$. The errors found are given in **Table 1**. By error, we mean error = | true value − approximate value |.

Table 1

x	$h = 0\cdot1$	$h = 0\cdot05$	$h = 0\cdot025$
0·1	$8\cdot34 \times 10^{-5}$	$2\cdot08 \times 10^{-5}$	$5\cdot19 \times 10^{-6}$
0·2	$1\cdot66 \times 10^{-4}$	$4\cdot11 \times 10^{-5}$	$1\cdot02 \times 10^{-5}$
0·3	$2\cdot44 \times 10^{-4}$	$6\cdot04 \times 10^{-5}$	$1\cdot50 \times 10^{-5}$
0·4	$3\cdot16 \times 10^{-4}$	$7\cdot79 \times 10^{-5}$	$1\cdot93 \times 10^{-5}$
0·5	$3\cdot80 \times 10^{-4}$	$9\cdot33 \times 10^{-5}$	$2\cdot31 \times 10^{-5}$
0·6	$4\cdot33 \times 10^{-4}$	$1\cdot06 \times 10^{-4}$	$2\cdot61 \times 10^{-5}$
0·7	$4\cdot74 \times 10^{-4}$	$1\cdot15 \times 10^{-4}$	$2\cdot83 \times 10^{-5}$
0·8	$4\cdot99 \times 10^{-4}$	$1\cdot21 \times 10^{-4}$	$2\cdot96 \times 10^{-5}$
0·9	$5\cdot09 \times 10^{-4}$	$1\cdot22 \times 10^{-4}$	$2\cdot98 \times 10^{-5}$
1·0	$5\cdot02 \times 10^{-4}$	$1\cdot19 \times 10^{-4}$	$2\cdot89 \times 10^{-5}$

Table 2

x	$h = 0\cdot1$	$h = 0\cdot05$	$h = 0\cdot025$
0·1	$5\cdot49 \times 10^{-4}$	$1\cdot35 \times 10^{-4}$	$3\cdot34 \times 10^{-5}$
0·2	$1\cdot18 \times 10^{-3}$	$2\cdot90 \times 10^{-4}$	$7\cdot18 \times 10^{-5}$
0·3	$1\cdot92 \times 10^{-3}$	$4\cdot75 \times 10^{-4}$	$1\cdot18 \times 10^{-4}$
0·4	$2\cdot85 \times 10^{-3}$	$7\cdot08 \times 10^{-4}$	$1\cdot76 \times 10^{-4}$
0·5	$4\cdot06 \times 10^{-3}$	$1\cdot01 \times 10^{-3}$	$2\cdot52 \times 10^{-4}$
0·6	$5\cdot69 \times 10^{-3}$	$1\cdot42 \times 10^{-3}$	$3\cdot56 \times 10^{-4}$
0·7	$7\cdot97 \times 10^{-3}$	$1\cdot99 \times 10^{-3}$	$5\cdot01 \times 10^{-4}$
0·8	$1\cdot12 \times 10^{-2}$	$2\cdot82 \times 10^{-3}$	$7\cdot07 \times 10^{-4}$
0·9	$1\cdot59 \times 10^{-2}$	$4\cdot02 \times 10^{-3}$	$1\cdot01 \times 10^{-3}$
1·0	$2\cdot28 \times 10^{-2}$	$5\cdot80 \times 10^{-3}$	$1\cdot46 \times 10^{-3}$

* *University of Wisconsin–Milwaukee, Milwaukee, Wisconsin.*

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Our second example is

$$y' = 1 + 2x - y + \int_0^x x(1 + 2x)e^{s(x-s)}y(s)ds.$$

The solution of the above problem is $y(x) = e^{x^2}$. The above equation was discussed by Pouzet (1960). The errors obtained are given in **Table 2.**

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Reference

Pouzet, P. (1960). "Méthode d'Intégration Numérique des Equations Intégrales et Integro-Différentielles du Type Volterra de Second Espèce Formules de Runge–Kutta," *Symposium on the Numerical Treatment of Ordinary Differential Equations, Integral and Integro-differential Equations*, Birkhauser Verlag, 1960, pp. 362–368.

Book Review

Quasilinearization & Nonlinear Boundary-Value Problems, by R. E. Bellman and R. E. Kalaba, 1965; 206 pages. (Barking: *Elsevier Publishing Company Ltd.*, 62s.)

The authors of this book observe that a problem is effectively solved if it has been reduced to an initial-value problem for a set of nonlinear simultaneous ordinary differential equations, since, provided that there are not more than a thousand equations, they may be accurately and rapidly integrated numerically, on large present-day electronic computers.

Chapter 1, "The Riccati Equation", is devoted to the important nonlinear equation: $v' + v^2 + p(t)u + q(t) = 0$ (with $v(0) = c$). This may be solved by taking an arbitrary function $v_0(t)$ (with $v_0(0) = c$) and then constructing the sequence of functions $v_n(t)$ ($n = 1, 2, \ldots$), by solving successively the *linear* differential equations: $v'_n + (2v_{n-1} + p(t))v_n + q(t) - v^2_{n-1} = 0$, with $v_n(0) = c$ (which can be solved analytically), as in the Newton-Raphson-Kantorovich approximation scheme. It is shown that the sequence $v_n(t)$ converges quadratically and monotonically to $v(t)$.

Chapter 2, on "Two-Point Boundary-Value Problems for Second-Order Differential Equations", treats the solution of *linear* second-order differential equations with two-point boundary conditions by integrating the equations for two sets of initial conditions (*e.g.* by a Runge-Kutta method), and then using linear interpolation between these two solutions so as to satisfy the given boundary condition at the end of the range. No warning is given about the numerical difficulties which often vitiate this procedure. Nonlinear equations are then solved by constructing a sequence of functions $u_n(x)$ (each satisfying the given two-point boundary conditions), with $u_0(x)$ arbitrary, and applying the technique of quasilinearization. For example, if the equation is $u'' = f(u)$, then at each stage of the iteration we solve the *linear* equation for $u_n(x)$ with two-point boundary conditions: $u''_n = f(u_{n-1}) + (u_n - u_{n-1})f'(u_{n-1})$. The sequence $u_n(x)$ converges quadratically to $u(x)$, whereas the Picard algorithm would give only linear convergence.

Chapter 3, on "Monotone Behaviour and Differential Inequalities", examines various conditions under which the sequence of functions generated by quasilinearization can be shown to converge monotonically. Chapter 4, on "Systems of Differential Equations, Storage and Differential Approximation", extends quasilinearization to systems of nonlinear differential equations with two-point (and multi-point) boundary conditions. Variations of the technique are discussed, in which the working space required in a computer may be economized at the expense of additional computing time.

Chapter 5, on "Partial Differential Equations", applies the technique of quasilinearization to non-linear second-order elliptic and parabolic partial differential equations. Chapter 6, on "Applications in Physics, Engineering and Biology" applies quasilinearization to a wide variety of problems, including the van der Pol equation, optimal design and control, periodogram analysis and analysis of the operation of the heart. FORTRAN programs for solving these problems are given as appendices.

Each chapter has a detailed bibliography, and there is an adequate index. Equation (2.16) is not quite correct, (19.4) on page 51 is misprinted, and a misprint in (4.3) on page 79 makes a numerical example incomprehensible. This book suggests a fruitful approach to a wide variety of problems involving non-linear differential equations, paying particular importance to rapid (quadratic) convergence to the solution, and also to monotonicity.

G. J. Tee

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