# A note on the asymptotic error constant of a certain method for solving equations 

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#### Abstract

A convenient recurrence relation is derived for determining the asymptotic error constant of a class of rational function iterative methods for solving equations.


## 1. Introduction

In a recent paper (Jarratt and Nudds, 1965), a method of solving equations was proposed in which a rational function of the form

$$
\begin{equation*}
y=\frac{x-a}{b_{m} x^{m}+b_{m-1} x^{m-1}+\ldots+b_{1} x+b_{0}} \tag{1.1}
\end{equation*}
$$

was fitted through $m+2$ points

$$
\left(x_{i}, f\left(x_{i}\right)\right), \quad i=n, n-1, \ldots, n-m-1
$$

the next approximation being given by $x_{n+1}=a$. It was shown that if the error, $\epsilon_{i}$, of the $i$ th estimate is defined by $x_{i}=\epsilon_{i}+\theta$, where $\theta$ is a root of the equation $f(x)=0$, then asymptotically the errors of the process derived from (1.1) by using at each stage the latest $m+2$ estimates, satisfy

$$
\begin{equation*}
\epsilon_{n+1}=A \prod_{i=0}^{m+1} \epsilon_{n-i} \tag{1.2}
\end{equation*}
$$

where $A$ is a constant depending on the values of the first $m+2$ derivatives of $f$ evaluated at $x=\theta$.

The purpose of this note is to obtain a method for deducing the actual form of $A$, which was not given in the original paper, and which is of importance for further theoretical and practical investigations.

## 2. Analysis

We first recapitulate briefly the analysis which led to (1.2). By fitting (1.1) to the points

$$
\left(x_{i}, f\left(x_{i}\right)\right), \quad i=n, n-1, \ldots, n-m-1
$$

we derive $m+2$ equations

$$
\begin{array}{r}
a+b_{0} f_{i}+b_{1} x_{i} f_{i}+\ldots+b_{m} x_{i}^{m} f_{i}=x_{i}, \quad i=n, n-1, \ldots \\
\ldots, n-m-1 \quad(2.1) \tag{2.1}
\end{array}
$$

together with the equation

$$
\begin{equation*}
a=x_{n+1} \tag{2.2}
\end{equation*}
$$

which is abtained by setting $y=0$ in (1.1) to predict the next approximation. (2.1) and (2.2) represent a set of $m+3$ equations in the $m+2$ coefficients $a, b_{i}, i=0$, $1, \ldots, m$, and hence for consistency the determinant of the set must vanish. Using this condition together with the substitution $x_{i}=\epsilon_{i}+\theta, \quad i=n+1, n, \ldots, n-m-1$,
and simplifying, we obtain the relation

$$
\left|\begin{array}{cccccc}
1 & \epsilon_{n+1} & 0 & 0 & \ldots & 0 \\
1 & \epsilon_{n} & f_{n} & \epsilon_{n} f_{n} & \ldots & \\
1 & \epsilon_{n-1} & f_{n-1} & \epsilon_{n-1} f_{n-1} & \ldots & \epsilon_{n}^{m} f_{n} \\
\cdot & \cdot & \cdot & \cdot & & \epsilon_{n-1}^{m} f_{n-1} \\
\cdot & \cdot & \cdot & \cdot & & \cdot \\
\cdot & \cdot & \cdot & \cdot & & \cdot \\
\cdot & \cdot & \cdot & \cdot & & \cdot \\
1 & \epsilon_{n-m-1} & f_{n-m-1} & \epsilon_{n-m-1} f_{n-m-1} & \epsilon_{n-m-1}^{m} f_{n-m-1}
\end{array}\right|
$$

from which

$$
\begin{equation*}
\epsilon_{n+1}=\left.\frac{\mid \boldsymbol{\epsilon}}{} \quad \mathbf{f} \quad \mathbf{\epsilon f} \ldots \boldsymbol{\epsilon}^{m} \mathbf{f}\right|_{n, n-m-1} \tag{2.4}
\end{equation*}
$$

where the notation of the paper referred to in the introduction has been used. Writing now $f\left(x_{i}\right)=\sum_{r=1}^{\infty} c_{r} \epsilon_{i}^{r}$ for $i=n, n-1, \ldots, n-m-1$, Where $c_{r}=f^{(r)}(\theta) / r$ ! and $c_{0}=f(\theta)=0$ and substituting in (2.4) we have

$$
\begin{align*}
\epsilon_{n+1}= & \prod_{i=0}^{m+1} \epsilon_{n-i} \\
& \times \frac{\left|\mathbf{1} \sum_{1}^{\infty} c_{r} \epsilon^{r-1} \sum_{1}^{\infty} c_{r} \epsilon^{r} \ldots \sum_{1}^{\infty} c_{r} \epsilon^{r+m-1}\right|_{n, n-m-1}}{\left|\mathbf{1} \sum_{1}^{\infty} c_{r} \epsilon^{r} \sum_{1}^{\infty} c_{r} \epsilon^{r+1} \ldots \sum_{1}^{\infty} c_{r} \epsilon^{r+m}\right|_{n, n-m-1}} \tag{2.5}
\end{align*}
$$

If we assume $c_{1} \neq 0$, as is the case for simple roots, it is easy to verify that the lowest-order terms in the development of each of the determinants depend on alternants of

$$
\begin{equation*}
\left|1 \in \epsilon^{2} \ldots \epsilon^{m+1}\right|_{n, n-m-1} \tag{2.6}
\end{equation*}
$$

Hence

$$
\epsilon_{n+1}=K_{m} \prod_{i=0}^{m+1} \epsilon_{n-i}\left[1+0\left(\hat{\epsilon}_{n, n-m-1}\right)\right]
$$

where

$$
\hat{\epsilon}_{n, n-m-1}=\operatorname{Max}\left\{\left|\epsilon_{n}\right|,\left|\epsilon_{n-1}\right|, \ldots,\left|\epsilon_{n-m-1}\right|\right\}
$$

and $K_{m}$ is the asymptotic error constant.

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## 3. A recursion for $\boldsymbol{K}_{\boldsymbol{m}}$

We consider first the denominator of (2.5), and it is readily seen that the coefficient of (2.6) in this case is simply $c_{1}^{m+1}$. For the expansion of the determinant in the numerator of (2.5), however, the pattern is more complicated as this expansion leads to a large number of non-zero determinants which are alternants of (2.6), and it is necessary to consider how the coefficient of each of these is formed. We begin by remarking that of the terms of the infinite series $\sum_{r=1}^{\infty} c_{r} \epsilon^{r+j-3}$ which occurs in the $j$ th column of the determinant

$$
\begin{equation*}
\left|1 \quad \sum_{1}^{\infty} c_{r} \boldsymbol{\epsilon}^{r-1} \sum_{1}^{\infty} c_{1} \boldsymbol{\epsilon}^{r} \ldots \sum_{1}^{\infty} c_{r} \boldsymbol{\varepsilon}^{r+m-1}\right|_{n, n-m-1} \tag{3.1}
\end{equation*}
$$

$$
j=3,4, \ldots, m+2
$$

only the first $m-j+4$ terms, namely $c_{1} \epsilon^{j-2}$, $c_{2} \mathbf{\epsilon}^{j-1}, \ldots, c_{m-j+4} \epsilon^{m+1}$, make any contribution to the set of lowest-order non-zero determinants obtained by expanding (3.1). In addition, the relevant terms of the series $\sum_{1}^{\infty} c_{r} \epsilon^{r-1}$ from the second column are $c_{2} \epsilon$, $c_{3} \epsilon^{2}, \ldots, c_{m+2} \epsilon^{m+1}$. It is now possible to see how many terms there will be in $K_{m}$. Starting with the last column for which $j=m+2$, we see that the only possibilities in this position are the terms $c_{1} \epsilon^{m}$ or $c_{2} \epsilon^{m+1}$. If we select one of these, then for the next column with $j=m+1$, for which there are basically three possibilities, $c_{1} \epsilon^{m-1}, c_{2} \epsilon^{m}$ or $c_{3} \epsilon^{m+1}$, there will be again just 2 choices, since we must not repeat the selection made for $j=m+2$, as this would lead to a zero determinant. Continuing in this way we find that for $j=m+2$, $m+1, \ldots, 4,3$ there are two possible choices available each time, leaving only one choice for $j=2$. Hence the number of terms in $K_{m}$ will be $2^{m}$. We now investigate the actual form of these terms, each of which consists, of course, of a coefficient multiplying an alternant of (2.6). Starting again at the last column with $j=m+2$, we select the highest order term, $c_{2} \epsilon^{m+1}$. In consequence no term containing $\boldsymbol{\epsilon}^{m+1}$ may be used in any of the other columns, and we see that the remaining possible permutations of terms in columns $2,3, \ldots, m+1$ are precisely those which would appear in the development of the lowest-order terms of the determinant

$$
\left|1 \sum_{1}^{\infty} c_{r} \epsilon^{r-1} \sum_{1}^{\infty} c_{r} \epsilon^{r} \ldots \sum_{1}^{\infty} c_{r} \epsilon^{r+m-3}\right|_{n, n-m}
$$

which is associated with $K_{m-1}$. Thus it is clear that those permutations arising from fixing the term $c_{2} \epsilon^{m+1}$ in the last column of (3.1) will lead to a coefficient of $c_{2} \cdot c_{1}^{m} K_{m-1}$ for the factor $\left|\mathbf{1} \in \boldsymbol{\epsilon}^{2} \ldots \epsilon^{m+1}\right|_{n, n-m-1}$.

We now move to the $(m+1)$ th column of (3.1) and select the term $c_{3} \epsilon^{m+1}$ to appear in this position. This forces the term $c_{1} \epsilon^{m}$ into the last column, and the permutations possible on columns $2,3, \ldots, m$ are now those associated with $K_{m-2}$. Hence we get a coefficient $c_{1} c_{3}$ $c_{1}{ }^{m-1} K_{m-2}$ attached to $\left|\mathbf{1} \in \epsilon^{2} \ldots \epsilon^{m} \epsilon^{m+1}\right|_{n, n-m-1}$. Continuing in this way by fixing the term in $\boldsymbol{\epsilon}^{m+1}$ in each of the columns in turn, we cover all the possible permutations of the terms which give rise to alternants of (2.6), and we derive ultimately the relation

$$
\begin{aligned}
K_{m}= & \frac{1}{c_{1}^{m+1}}\left\{c_{1}^{m} \cdot c_{2} K_{m-1}-c_{1}^{m-1} \cdot c_{1} c_{3} K_{m-2}\right. \\
& +c_{1}^{m-2} \cdot c_{1}^{2} c_{4} \cdot K_{m-3}-\ldots \\
& \left.+(-1)^{m-1} c_{1}^{m} \cdot c_{m+1} K_{0}+(-1)^{m} c_{1}^{m} c_{m+2}\right\}
\end{aligned}
$$

giving

$$
\begin{align*}
K_{m}= & \frac{1}{c_{1}}\left\{c_{2} K_{m-1}-c_{3} K_{m-2}+c_{4} K_{m-3}-\ldots\right. \\
& \left.+(-1)^{m-1} c_{m+1} K_{0}+(-1)^{m} c_{m+2}\right\} \tag{3.2}
\end{align*}
$$

Thus by determining $K_{0}$ we shall have a convenient method of generating recursively the asymptotic error constant of any order of rational iteration function derived from fitting a formula of the type (1.1). Moreover $K_{0}$ is the error constant obtained from fitting $y=\frac{x-a}{b}$, which leads to the well known secant iteration for which

$$
\begin{equation*}
K_{0}=\frac{c_{2}}{c_{1}} \tag{3.3}
\end{equation*}
$$

(3.2) and (3.3) have been applied to work out the first few asymptotic error constants and these are given in Table 1.

Table 1

| $m$ | $K_{m}$ |
| :---: | :---: |
| 0 | $c_{2} / c_{1}$ |
| 1 | $-c_{3} / c_{1}+c_{2}^{2} / c_{1}^{2}$ |
| 2 | $c_{4} / c_{1}-2 c_{2} c_{3} / c_{1}^{2}+c_{2}^{3} / c_{1}^{3}$ |
| 3 | $-c_{5} / c_{1}+2 c_{2} c_{4} / c_{1}^{2}-3 c_{2}^{2} c_{3} / c_{1}^{3}+c_{3}^{2} / c_{1}^{2}+c_{2}^{4} / c_{1}^{4}$ |

It is interesting to note that the expression for $m=1$ corresponding to the method of linear fractions is in agreement with that given in the paper by Jarratt and Nudds (1965).

## Reference

Jarratt, P., and Nudds, D. (1965). "The use of rational functions in the iterative solution of equations on a digital computer", The Computer Journal, Vol. 8, pp.62-65.


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