The stability of predictor-corrector methods

By G. Hall*

This paper introduces a new method for finding the range of absolute stability for predictor-correctors. An example is given and the method is compared with that in common use. Some numerical results for a particular class of predictor-correctors are included.

1. Introduction

A general predictor-corrector pair may be written

$$y_n = \sum_{i=1}^{k} a_i^* y_{n-i} + h \sum_{i=1}^{k+1} b_i^* y'_{n-i} \qquad (1)$$

$$y_n = \sum_{i=1}^{k} a_i y_{n-i} + h \sum_{i=0}^{k} b_i y'_{n-i} \qquad (2)$$

where y_p is a calculated value for $y(x_0 + ph)$, the true solution of the differential equation $y' = f(x, y)$, $y(x_0) = \eta$. Both expressions for y_n involve k values of y, $k + 1$ values of y' and $2k + 1$ constant (possibly zero, except for b_0) multipliers; h is the step-length of the integration.

The corrector, (2), involves the term

$$hb_0 y'_n \; (= hb_0 f(x_n, y_n)).$$

It may be solved for y_n by the following iteration, which will converge if $\left| hb_0 \frac{\partial f}{\partial y} \right| < 1$ (see Henrici, 1962, pp. 215–217). First calculate y_n from (1), the predictor, and then $y'_n = f(x_n, y_n)$. Now iterate with (2), calculating y_n using the most recent approximation to y'_n on the right-hand side and then calculating y'_n again.

We assume here that this iteration with the corrector is done a fixed number of times, m, for each step of the integration. We may choose to stop at an evaluation of y_n, C, or an evaluation of y'_n from $y'_n = f(x_n, y_n)$, E. These alternatives are represented by $P(EC)^m$ and $PE(CE)^m$ respectively, where P is the first evaluation of y_n from the predictor.

The polynomial equations governing the stability of a predictor-corrector pair when used in either of the above ways were obtained by Hull and Creemer (1963). For $P(EC)^m$ it is†

$$s^{k+1}C(s) + \theta^{m-1}\bar{h}\left(C(s) \sum_{i=1}^{k+1} B_i s^{k+1-i} + \left(\sum_{i=0}^{k} b_i s^{k-i} \right) \sum_{i=1}^{k+1} (A_i + \bar{h}B_i) s^{k+1-i} \right) = 0 \qquad (3)$$

and for $PE(CE)^m$

$$sC(s) + \theta^m \sum_{i=1}^{k+1} (A_i + \bar{h}B_i) s^{k+1-i} = 0 \qquad (4)$$

† The first differs from that given in Hull and Creemer which is believed to be in error.

* *Mathematics Department, The University, Manchester,* 13.

where

$$C(s) = s^k - \sum_{i=0}^{k} (a_i + \bar{h}b_i) s^{k-i}$$

$$\bar{h} = h \frac{\partial f}{\partial y}$$

$$\theta = \bar{h} b_0$$

$$\alpha = \theta - 1$$

$$A_i = a_i + \alpha a_i^*$$

and

$$B_i = b_i + \alpha b_i^*.$$

These are the characteristic equations of a system of linear difference equations satisfied by the error $e_n = y_n - y(x_0 + nh)$. If the roots of (3), for example, are $s_i (i = 1, 2, \ldots, 2k + 1)$ then e_n is of the form

$$e_n = \sum_{i=1}^{2k+1} K_i (s_i)^n$$

where the K_i are constants. For a single error, after it is made, we say that we have absolute stability when $|s_i| \leqslant 1$ (all i) and relative stability when $|s_i| \leqslant e^{\bar{h}}$ (all i); in each case zeros equal to the bounding values must be simple. Relative stability ensures that any error introduced does not grow more rapidly than the solution, whereas absolute stability ensures that any such error will not grow larger.

Chase (1962) has examined the stability of some predictor-corrector pairs. His method was to solve the stability equation for a particular value of m and a range of values of $\bar{h}$, thus finding the range of $\bar{h}$ in which the algorithm is stable. This method was used by Brown, Riley and Bennet (1965) who considered some ways in which the pair

$$y_n = y_{n-1} + \frac{h}{24}(55y'_{n-1} - 59y'_{n-2} + 37y'_{n-3} - 9y'_{n-4}) \qquad (5)$$

$$y_n = y_{n-1} + \frac{h}{24}(9y'_n + 19y'_{n-1} - 5y'_{n-2} + y'_{n-3}) \qquad (6)$$

may be used. This predictor-corrector pair is a particular example of an Adams–Bashforth predictor coupled with a Moulton corrector. Such pairs exist of any order, (see Henrici, 1962, pp. 191–199). We deal exclusively with this class of methods here, although the method to be described for finding the bounds of absolute stability may be applied to any predictor-corrector pair.

2. Analysis of stability for $P(EC)^m$

Consider the polynomial equation (3), arising from the algorithm $P(EC)^m$, which is of degree $2k+1$. For the special class of methods under consideration it simplifies to

$$S^k \sum_{i=0}^{k+1} c_i s^{k+1-i} = 0 \quad (c_{k+1} \neq 0). \tag{7}$$

The k zero roots have no effect on stability so we consider

$$\sum_{i=0}^{k+1} c_i s^{k+1-i} = 0. \tag{8}$$

The coefficients are

$$c_0 = 1 - \bar{h}b_0$$

$$c_1 = -1 - \bar{h}b_1 + \theta^{m-1}\bar{h}(b_1 + \alpha b_i^* + \bar{h}b_0^2)$$

$$c_j = -\bar{h}b_j + \theta^{m-1}\bar{h}(b_j + \alpha b_j^* - b_{j-1} - \alpha b_{j-1}^* + \bar{h}b_0 b_{j-1})$$

$$j = 2, 3, \ldots, k+1.$$

To equation (8) apply the transformation $s = \dfrac{1+z}{1-z}$ which maps the interior of the unit circle onto the left half-plane, the unit circle onto the imaginary axis. The transformed equation is

$$\sum_{i=0}^{k+1} v_i z^{k+1-i} = 0 \tag{9}$$

where the coefficients v_i are linear combinations of the c_i.

It may be easily shown that a necessary condition that the roots of (9) lie in the left half-plane is that the coefficients v_i have the same sign (see Henrici, p. 230). One zero of the original stability polynomial (8), approximates $e^{\bar{h}}$ (corresponding to the solution of the differential equation); therefore, we can only look for absolute stability when $\bar{h} \leqslant 0$; that is when $\theta \leqslant 0$, since $b_0 > 0$ for the class of methods under consideration. Imposing the condition $|\theta| < 1$ so that the corrector iterations would converge, we try to find in what part of the range $-1 < \theta \leqslant 0$ the algorithm is absolutely stable. The predictor-correctors under consideration are all absolutely stable as $\theta \to -0$. We wish to find the value θ^L of θ in $(-1, 0)$, if any, at which the algorithm violates absolute stability; $(\theta^L, 0)$ is then the range of absolute stability.

After making the above transformation we find that the coefficients v_i take a particularly simple form; one sees immediately that $v_i > 0$ ($i \neq 0$, $-1 < \theta < 0$) and that therefore a necessary condition for absolute stability is that $v_0 > 0$. We find

$$v_0 = c_0 - c_1 + c_2 - \ldots + (-1)^{k+1}c_{k+1} \tag{10}$$

$$= \frac{1}{b_0}(2b_0 - \tau\theta - \theta^m(2(b_0 - \tau - \tau^*) + \theta(2\tau^* + \tau)))$$

where

$$\tau = b_0 - b_1 + b_2 - \ldots + (-1)^k b_k$$

$$\tau^* = b_1^* - b_2^* + b_3^* - \ldots + (-1)^k b_{k+1}^*.$$

For m even, $v_0 > 0$ for all θ in $(-1, 0)$. When m is odd $v_0 > 0$ for all θ in $(\theta^L, 0)$ where $\theta^L > -1$; θ^L is a root of $v_0 = 0$ and is conjectured to be the lower bound of absolute stability.

To prove this we have to consider what happens in the range of $v_0 > 0$. This is only a necessary condition for absolute stability so even when it is satisfied there may still be a root in the right half-plane. However, as $\theta \to -0$ the algorithm becomes stable. If such a root exists it must cross the imaginary axis at a value θ^s of θ. It is possible to calculate this value θ^s. When m is even we find that there is such a point in $(-1, 0)$, but not when m is odd. This is opposite behaviour to that of θ^L—which, being a root of $v_0 = 0$, corresponds to the case of the last root entering the left half-plane at infinity. When m is even $(\theta^s, 0)$ is the range of absolute stability, for in this case $v_0 > 0$ in $(-1, 0)$. When m is odd $(\theta^L, 0)$ is the range of absolute stability, there being no value θ^s in $(-1, 0)$.

θ^s may be found by obtaining the condition that (9) have a root $z = pi$ (p real). This condition is given in its general form in Section 6. It reduces to a polynomial equation in θ,

$$u(\theta) = 0 \tag{11}$$

whose degree depends on m.

When we have obtained the polynomials $v_0(\theta)$ and $u(\theta)$ we calculate the range of stability by solving one or the other for its root in $(-1, 0)$.

3. Example

Here are given the details of the case $k = 1$

$$y_n = y_{n-1} + \frac{h}{2}(3y_{n-1} - y_{n-2})$$

$$y_n = y_{n-1} + \frac{h}{2}(y_n + y_{n-1}).$$

We find that

$$v_0 = 1 + \theta^m(3 - 4\theta)$$

$$v_1 = (2 - \theta)(1 - \theta^m)$$

$$v_2 = (1 - 2\theta)(1 - \theta^m)$$

$$v_3 = -\theta(1 - \theta^m).$$

It is obvious that v_1, v_2, v_3 are positive in $(-1, 0)$. When m is odd we solve

$$v_0 = 1 + \theta^m(3 - 4\theta) = 0$$

and when m is even

$$u(\theta) = 0$$

which in this case is simply

$$v_1 v_2 = v_0 v_3$$

Table 1

Lower stability bounds for $P(EC)^m$

$k \backslash m$	1	2	3
0	$-0{\cdot}67$	$-1{\cdot}00$	$-0{\cdot}86$
1	$-0{\cdot}25$	$-0{\cdot}74$	$-0{\cdot}57$
2	$-0{\cdot}12$	$-0{\cdot}49$	$-0{\cdot}43$
3	$-0{\cdot}06$	$-0{\cdot}33$	$-0{\cdot}33$

Table 2

Lower stability bounds for $PE(CE)^m$

$k \backslash m$	0	1	2
0	$-1{\cdot}00$	$-1{\cdot}00$	$-1{\cdot}00$
1	$-0{\cdot}50$	$-1{\cdot}00$	$-0{\cdot}74$
2	$-0{\cdot}23$	$-0{\cdot}72$	$-0{\cdot}53$
3	$-0{\cdot}11$	$-0{\cdot}48$	$-0{\cdot}40$

or
$$(1-\theta)-\theta^m(1-3\theta)=0.$$

For $m=1$ $\qquad \theta^L = -0{\cdot}25.$

For $m=2$ $\qquad \theta^s = -0{\cdot}74.$

4. The case $PE(CE)^m$

The development of the algorithm $PE(CE)^m$ from equation (4) is similar to the above. The equivalent expression to (10) is

$$v_0 = \frac{1}{b_0}(2b_0 - \tau\theta - \theta^{m+1}(2b_0 - \tau - \tau^* + \tau^*\theta)). \quad (12)$$

Numerical results for both $P(EC)^m$ and $PE(CE)^m$ are below.

5. Conclusions

The method of Chase (1962) involves solving the stability equation several times, for all its roots, for each value of m. Here one need only solve once, for a particular zero of either $v_0(\theta)$ or $u(\theta)$, for each value of m. Our main difficulty is in obtaining the polynomial $u(\theta)$; for high order methods the manipulation becomes very tedious. The analysis in Sections 3 and 4 has been done for predictor-correctors of order up to that of the pair (5), (6). Of course one has the expressions for v_0, (10) and (12), for methods of any order. It is likely that these will give the stability bounds for m odd, although one needs to find the corresponding polynomials $u(\theta)$ to be sure of this.

It is apparent from the results of Section (7) that the algorithm $PE(CE)^m$ has a larger range of stability than $P(EC)^m$, when comparing bounds which involve the same number of evaluations, E.

The condition that equation (9) have a root $z = pi$ (p real) is either

$$\begin{vmatrix} v_1 & v_0 & 0 & 0 & \ldots 0 & 0 \\ v_3 & v_2 & v_1 & v_0 & \ldots 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ v_{k-1} & v_{k-2} & v_{k-3} & v_{k-4} & \ldots v_1 & v_0 \\ v_{k+1} & v_k & v_{k-1} & v_{k-2} & \ldots v_3 & v_2 \\ 0 & 0 & v_{k+1} & v_k & \ldots v_5 & v_4 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \ldots v_{k+1} & v_k \end{vmatrix} = 0 \ (k \text{ even})$$

or

$$\begin{vmatrix} v_1 & v_0 & 0 & 0 & \ldots 0 \\ v_3 & v_2 & v_1 & v_0 & \ldots 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ v_{k-2} & v_{k-3} & v_{k-4} & v_{k-5} & \ldots 0 \\ v_k & v_{k-1} & v_{k-2} & v_{k-3} & \ldots v_1 \\ 0 & v_{k+1} & v_k & v_{k-1} & \ldots v_3 \\ 0 & 0 & 0 & v_{k+1} & \ldots v_5 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \ldots v_k \end{vmatrix} = 0 \ (k \text{ odd}).$$

Tables 1 and **2** give the values of θ calculated as the lower bounds of absolute stability for the predictor-corrector methods we have considered. The corresponding value of h may be calculated from $b_0 h = \theta$.

I am indebted to Dr. D. Evans, director of Sheffield University Computer Laboratory, for supervising my work and to R. A. Brooker, of the Manchester Computer Laboratory, for help during the preparation of this paper.

References

Henrici, P. (1962). *Discrete Variable Methods in Ordinary Differential Equations*, Wiley & Sons, New York.

Hull, T. E., and Creemer, A. L. (1963). "Efficiency of Predictor-Corrector Procedures", *J. ACM*, Vol. 10, pp. 291–301.

Chase, P. E. (1962). "Stability Properties of Predictor-Corrector Methods for Ordinary Differential Equations", *J. ACM*, Vol. 9, pp. 457–468.

Brown, R. R., Riley, J. D., and Bennet, M. M. (1965). "Stability Properties of Adams–Moulton Type Methods", *Math. Comp.*, Vol. 19, pp. 90–96.